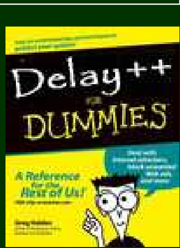
In delay there lies no plenty

Time-delay systems are also called systems with after effect or dead-time, hereditary systems, equations with deviating argument or differential-difference equations. They belong to the class of functional differential equations which are infinite dimensional, as opposed to ordinary differential equations. In spite of their complexity, they may often appear as simple infinite-dimensional models in the playground of partial differential equations. After the presentation of some motivating examples, the talk will try to show main differences arising from the presence of deviating time-arguments in the dynamics, seen from different points of view : state, solutions, stability, identification...

In delay there lies no plenty†



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INTERNATIONAL SCHOOL ON COMPLEX DYNAMICS ENGINEERING
BATZ-SUR-MER, OCTOBER 17-21, 2011
GDR 2984 DYCOEC – DYnamique et COntrôle des Ensembles Complexes

What's to come is still unsure:
In delay there lies no plenty;
Then come kiss me, sweet and twenty,
Youth's a stuff will not endure.

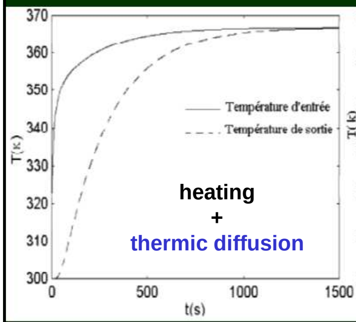
† Shakespeare, W. *Twelfe Night, Or what you will*, 2(3), 1599.

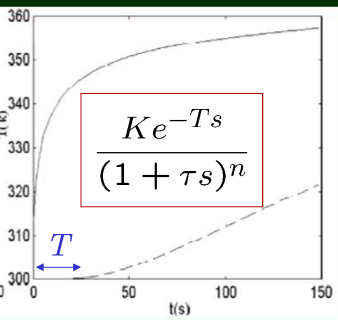
Delays : Classical examples

Strejć/Broïda – like models for industrial control

- frequent in process engineering
- simple and generic approximation
- PID controler ? ... poor results if $T > \tau$

➔ Smith predictor or « GPID » ...

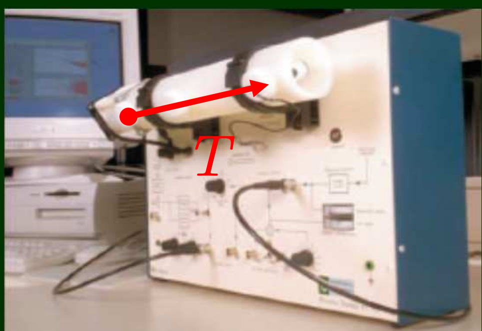


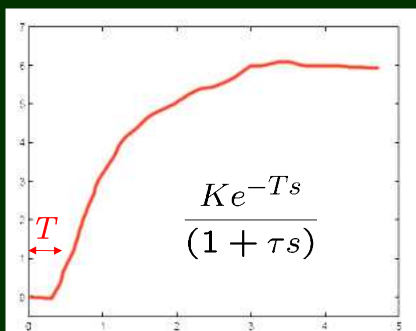


Ex: Steam Genererator LAGIS

Delays : Another classical example

... a great standard of control classes (Feedback)





... and of systems with transport phenomena

$$\left.\begin{array}{l} T \approx 20.10^{-3} sec, \\ \tau \approx 1 \;\; sec. \end{array}\right\} \text{PID ok}$$

PHYSICAL REVIEW E **79**, 026208 (2009)

Routes to chaos and multiple time scale dynamics in broadband bandpass nonlinear delay electro-optic oscillators

Michael Peil,[1] Maxime Jacquot,[1] Yanne Kouomou Chembo,[1] Laurent Larger,[1] and Thomas Erneux[2]

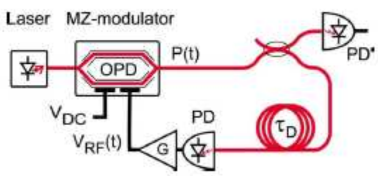


FIG. 1. (Color online) Experimental setup of the nonlinear electro-optic delay oscillator.

$$\tau x' = -x + f[x(t-\tau_D)]$$

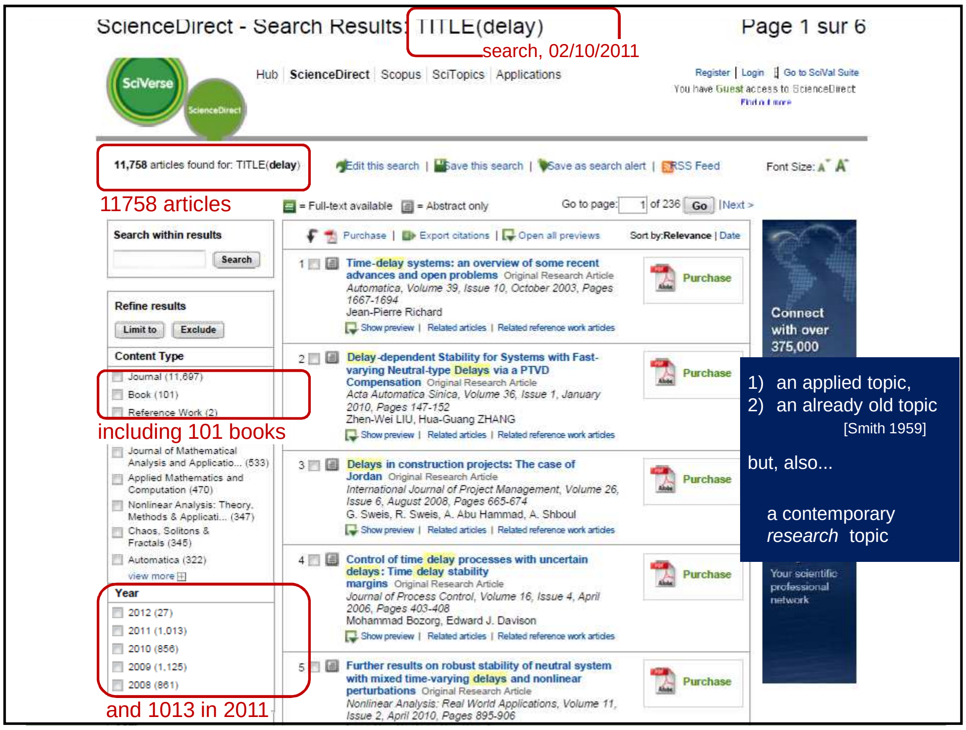
$$\theta^{-1}\int_{t_0}^{t} x(\eta)d\eta + \tau x' = -\left(1+\frac{\tau}{\theta}\right)x + f[x(t-\tau_D)]$$

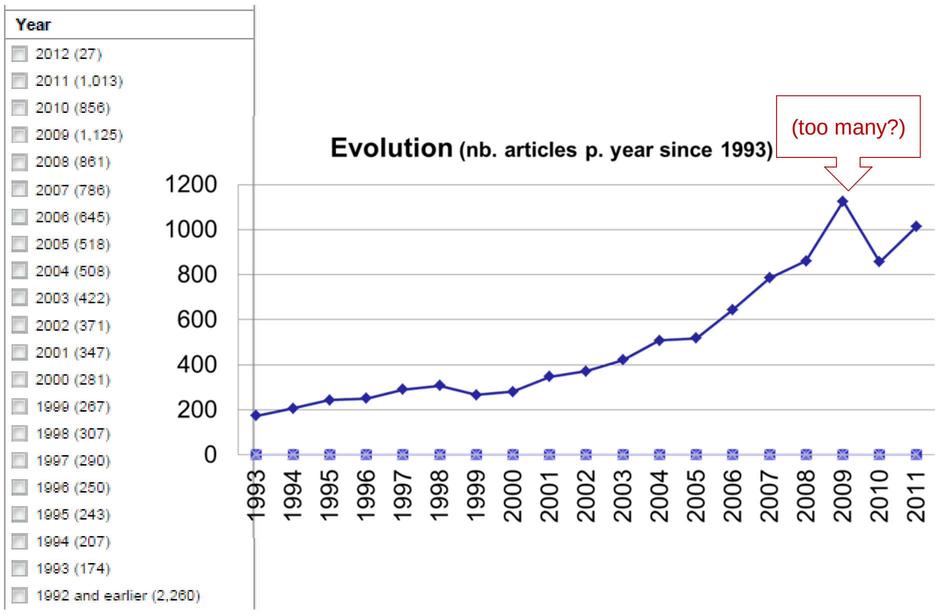
$$\theta \simeq 10^3 \tau_D \simeq 10^6 \tau.$$

delay τ_D of the order of 100 ns.

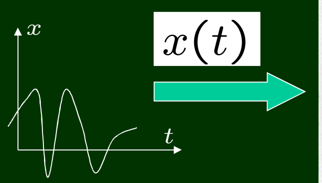
can be widely adjusted between θ and τ. For short delays, the accessible range is limited by the minimal length of the fibers and electrical connectors required to close the optoelectronic feedback loop. This length corresponds to about one meter or less of optical fiber, leading to a delay of a few nanoseconds or even tenths of nanoseconds. On the other hand, very long delay of several tens of microseconds, corresponding to several kilometers of optical fiber, can be realized owing to the low absorption of modern fibers. In this

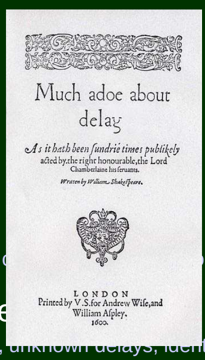
(a smarter system with transport phenomenon)

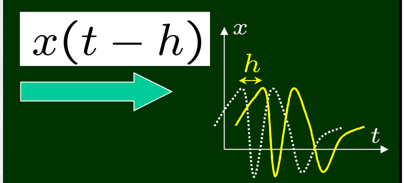




Much a do about delay ?

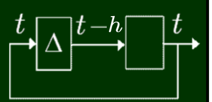






- applied problem

  engineering (NCS, telecom, ... gy, populations, nuclear...



- still open in many case

  closed-loop, variable delays, unknown delays, identification...

- the « simplest » infinite dimension problem

  functional equations, particular case of PDEs

- surprising properties

  damaging/improving by adding delays, data-sampling model...

Contents

Distinctive features of TDS?

Illustrative examples

- 1rst example (remote crtl.) → basic notions (stability, state, inf. dim.) *goodies: chaos*
- 2nd example : variable delay → counter-example
- 3rd example : sampling → delay for modelling ZoH
- 4th example : **Networked Control System** (master-slave)

Cauchy's problem for TDS

- Notion of solution
- Lipschitz-type condition
- And if the delay can vanish…

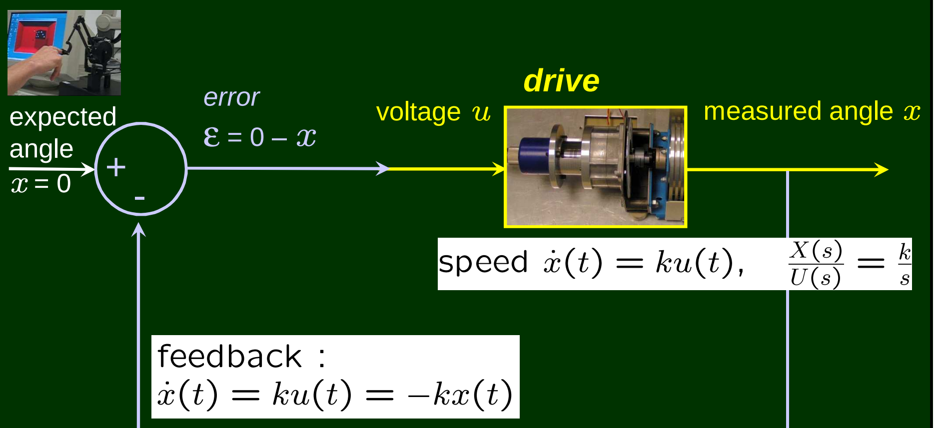
Stability and Lyapunov

- the LTI case
- 1rst Lyapunov (small states)
- small delays
- 2nd Lyapunov

Some words about identification

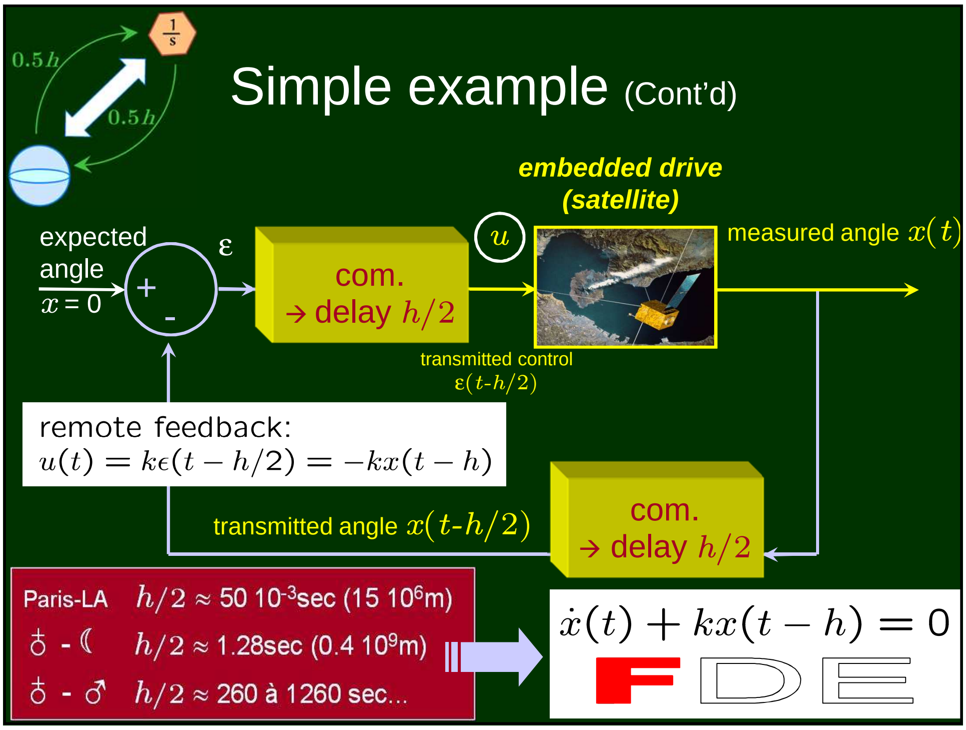
~~Mathematics~~

A simple example

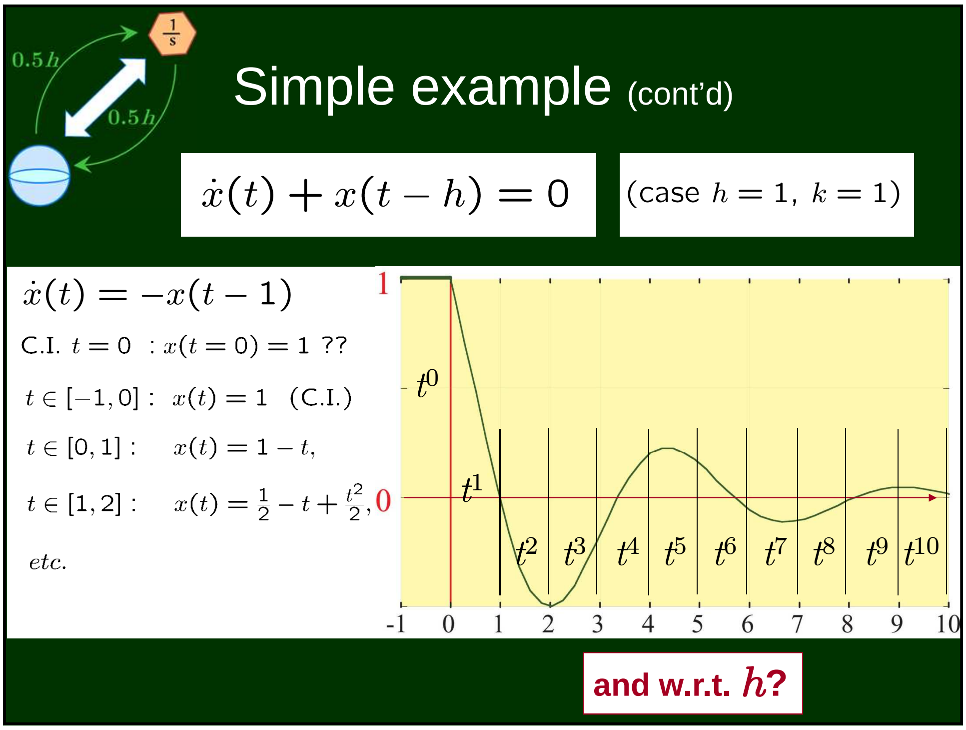


$$\dot{x}(t) + kx(t) = 0$$

ODE



Simple example (Cont'd)
0.5 h
0.5 h
1/s
embedded drive
(satellite)
expected angle
x = 0
+
-
ε
com.
→ delay h/2
u
measured angle x(t)
transmitted control
ε(t-h/2)
remote feedback:
$u(t) = k\epsilon(t - h/2) = -kx(t - h)$
transmitted angle x(t-h/2)
com.
→ delay h/2
Paris-LA h/2 ≈ 50 10^-3 sec (15 10^6 m)
♁ - ☾ h/2 ≈ 1.28sec (0.4 10^9 m)
♁ - ♂ h/2 ≈ 260 à 1260 sec...
$\dot{x}(t) + kx(t - h) = 0$
FDE



Simple example (cont'd)
0.5 h
0.5 h
1/s
$\dot{x}(t) + x(t - h) = 0$
(case $h = 1,\ k = 1$)
$\dot{x}(t) = -x(t - 1)$
C.I. $t = 0$: $x(t = 0) = 1$??
$t \in [-1, 0]$: $x(t) = 1$ (C.I.)
$t \in [0, 1]$: $x(t) = 1 - t,$
$t \in [1, 2]$: $x(t) = \frac{1}{2} - t + \frac{t^2}{2},$
etc.
1
0
t^0 t^1 t^2 t^3 t^4 t^5 t^6 t^7 t^8 t^9 t^{10}
-1 0 1 2 3 4 5 6 7 8 9 10
and w.r.t. h?

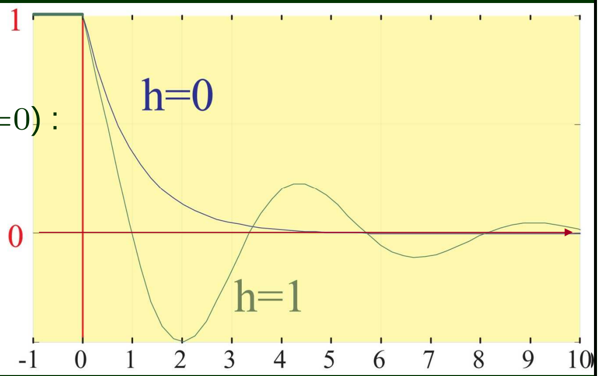
Simple example (cont'd)

$\dot{x}(t) + x(t-h) = 0$ w.r.t. h ?

$\dot{x}(t) + x(t-1) = 0$

to be compared with (h=0) :

$\dot{x}(t) + x(t) = 0$

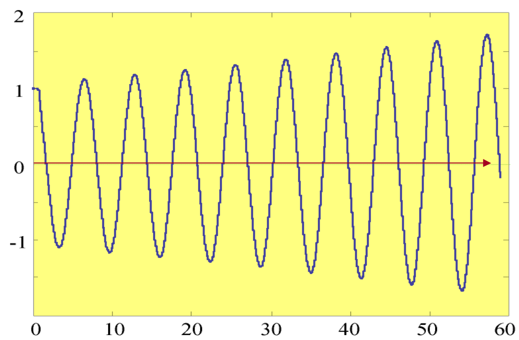


Simple example (cont'd)

$\dot{x}(t) + x(t-h) = 0$ $h = 1.6$

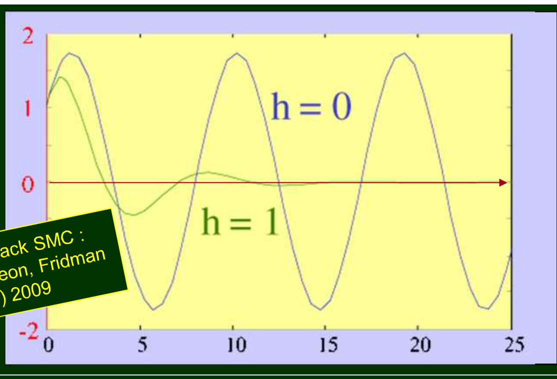
$\dot{x}(t) + x(t-1.6) = 0$

$t \in [-1, 0]$: $x(t) = 1$ (same I.C.)



(intermezzo...)

$$\ddot{y}(t) + y(t) - \tfrac{1}{2}y(t-h) = 0$$



see also output feedback SMC : Seuret, Edward, Spurgeon, Fridman IEEE TAC 54(2) 2009

a delay can have a stabilizing effect as well

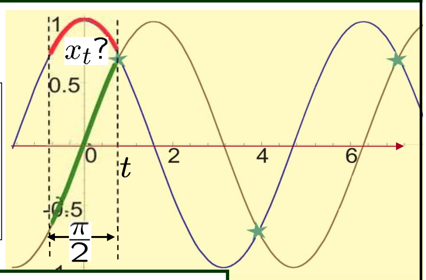
here, derivative effect: $y(t-h) \approx y(t) - h\dot{y}(t)$

Simple example (cont'd)

$$\dot{x}(t) + x(t-h) = 0 \qquad h = \tfrac{\pi}{2}$$

(Shimanov's notation , 1960)

$$\begin{aligned} \dot{x}(t) &= f\,(x_t, t, u_t)\,, & t &\geq t_0, \\ x_t(\theta) &= x(t+\theta), & -h &\leq \theta \leq 0, \\ u_t(\theta) &= u(t+\theta), & -h &\leq \theta \leq 0, \\ x(\theta) &= \varphi(\theta), & t_0 - h &\leq \theta \leq t_0, \end{aligned}$$



→ « state » notion ?

a variable $\phi(t)$ generating a unique solution starting at instant t

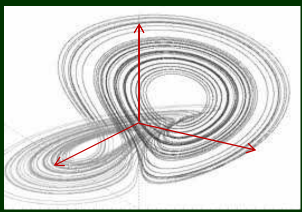
function x_t = state at time t

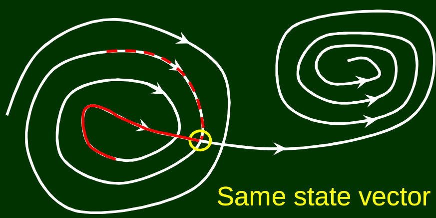
vector $x(t) = x_t(0)$ solution at t

function x_t ⇒ FDE ⇒ **infinite dim. syst.**

A note about chaos

- ODEs: no chaos for differential order < 3
- FDEs: possible for n=2…or less?





$$\dot{x}(t) = -\sin x(t-\tau)$$

$$\dot{x}_0 = \sin x_N,$$
$$\dot{x}_i = N(x_{i-1} - x_i)/\tau,$$
where $1 \leqslant i \leqslant N \to \infty$.

more than simulation →

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www.elsevier.com/locate/pla

A simple chaotic delay differential equation

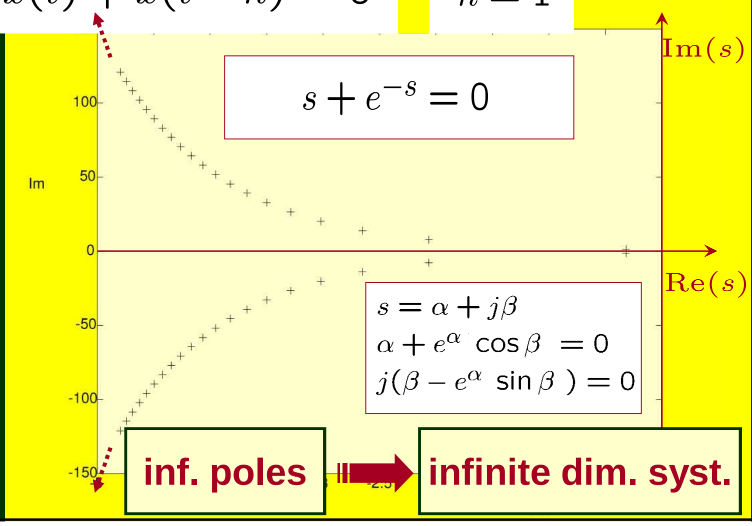
J.C. Sprott *

Department of Physics, University of Wisconsin–Madison, Madison, WI 53706, USA

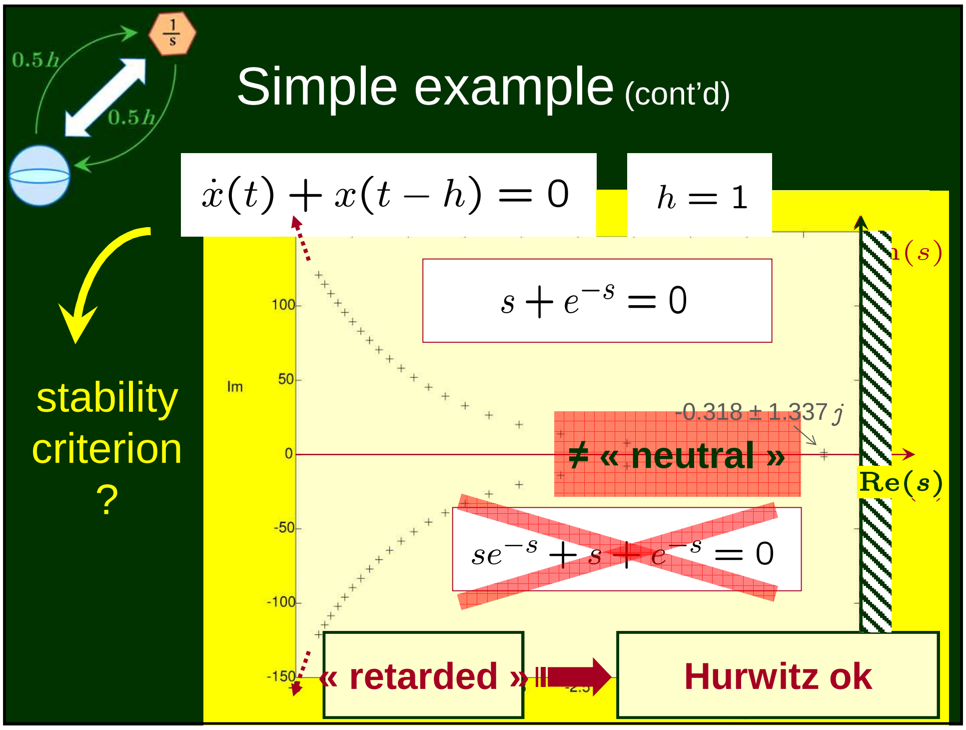
Simple example (cont'd)

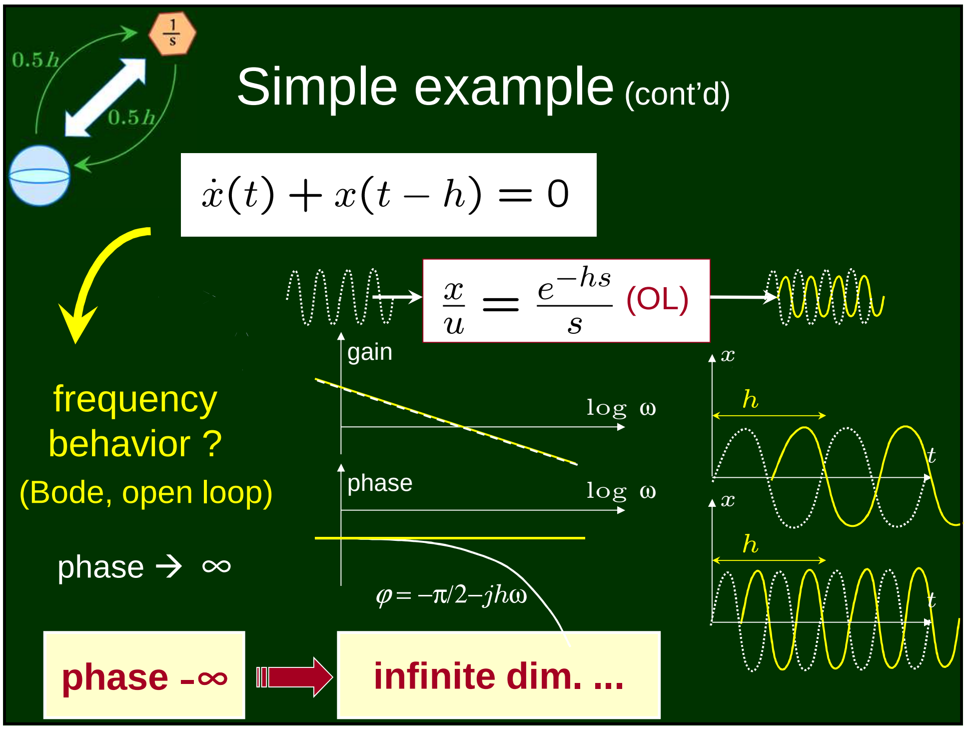
$$\dot{x}(t) + x(t-h) = 0 \qquad h = 1$$

poles?



inf. poles ⇒ **infinite dim. syst.**


Simple example (cont'd)
$\dot{x}(t) + x(t-h) = 0$
$h = 1$
$s + e^{-s} = 0$
stability criterion ?
Im
100
50
0
-50
-100
-150
-0.318 ± 1.337 j
≠ « neutral »
Re(s)
$se^{-s} + s + e^{-s} = 0$
« retarded »
Hurwitz ok


Simple example (cont'd)
$\dot{x}(t) + x(t-h) = 0$
$\frac{x}{u} = \frac{e^{-hs}}{s}$ (OL)
frequency behavior ?
(Bode, open loop)
gain
log ω
phase
log ω
$\varphi = -\pi/2 - jh\omega$
phase → ∞
phase -∞
infinite dim. ...
x
h
t
x
h
t

Si[mple example]

$\dot{x}(t) + x(t-h) = 0$

« state description » ?

$N \rightarrow \infty$

infinite dim...

$$X(t) = \begin{pmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ x(t-\theta) \\ x(t-2\theta) \\ \vdots \\ x(t-h) \end{pmatrix}, \text{ with } \theta = \frac{h}{N} \text{ and } N \to \infty,$$

$$\dot{X}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{x}(t-\theta) \\ \dot{x}(t-2\theta) \\ \vdots \\ \dot{x}(t-N\theta) \end{pmatrix} = \begin{pmatrix} -x_N(t) \\ \frac{N}{h}[x_0(t)-x_1(t)] \\ \frac{N}{h}[x_1(t)-x_2(t)] \\ \vdots \\ \frac{N}{h}[x_{N-1}(t)-x_N(t)] \end{pmatrix} = AX(t),$$

$$A = \begin{bmatrix} 0 & 0 & 0 & .. & -1 \\ \frac{N}{h} & -\frac{N}{h} & 0 & .. & 0 \\ 0 & \frac{N}{h} & -\frac{N}{h} & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & & 0 & \frac{N}{h} & -\frac{N}{h} \end{bmatrix} \in R^{\infty\times\infty}.$$

Simple example (the end!)

$$\dot{x}(t) + x(t-h) = 0$$

to sum up...

delay $\Rightarrow$
- strong influence on stability
- fonctionnal state
- infinite nb of poles (Hurwitz OK, Routh no)
- strong phase displacement ($\rightarrow -\infty$)

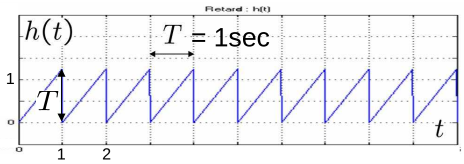
and, up to now, it was not that complicated
- constant delay
- linear, scalar system « 1rst order »

is it the same for variable delays $h(t)$?

a counter-example...

(counter-)example with variable delay

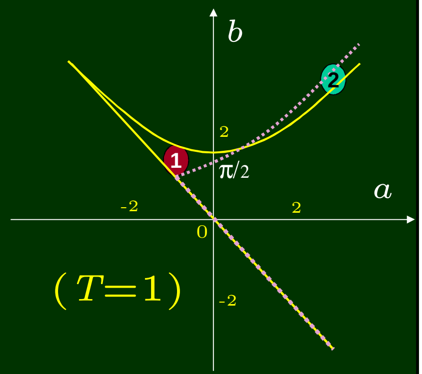
$$\dot{x}(t) = -ax(t) - bx(t - h(t)) \qquad (1)$$
$$h(t) = t - kT \quad \text{for} \quad kT < t \leq (k+1)T$$



is asymptotically stable iff (yellow zone) :

$$\left|(1 + \tfrac{b}{a})e^{-aT} - \tfrac{b}{a})\right| < 1 \qquad \text{si } a \neq 0$$
$$|1 - bT| < 1 \qquad \text{si } a = 0$$

and, for $h =$ cst ∊ [0,1] iff (pink zone)



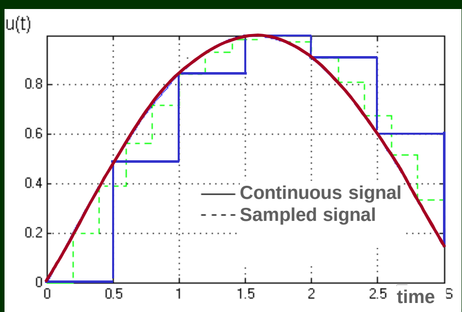
1 stable $h(t)$<1 - unstable h=cst<1

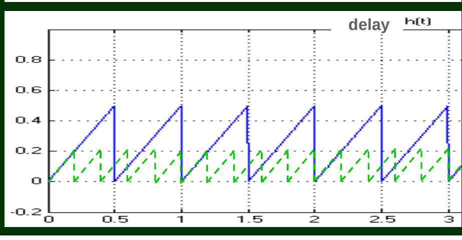
2 unstable $h(t)$<1 - stable h=cst<1

OK, but does such a delay $h(t)$ happen? another example...

Sampled systems: an interesting idea…

Mikheev et al. 88, Sobolev et al. 89. Aström et al. 92





Fridman, Seuret, Richard - *Automatica* 2004

Sampled and hold signal

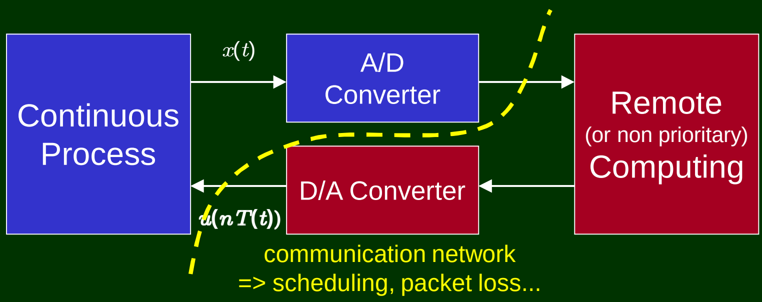
(depicted for a constant period)

⇕

Delayed signal with variable $h(t)$

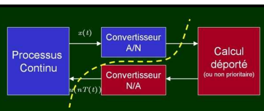
$$u(t) = u_d(t_k) = u_d(t - [t - t_k]) = u(t - h(t))$$

…with application to aperiodic sampling

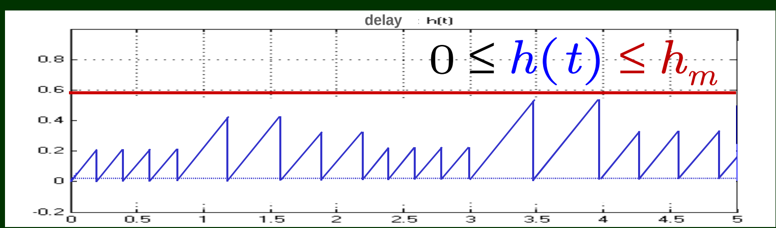


$$u(t) = g(x(nT)$$

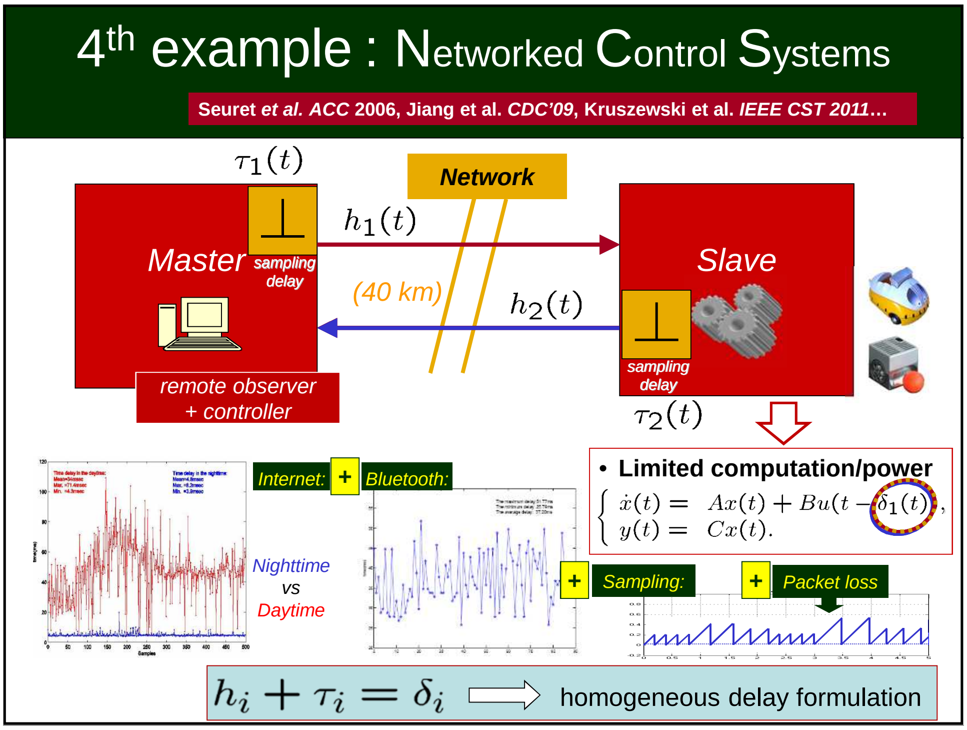
$$u(t) = g(x(nT(t))$$

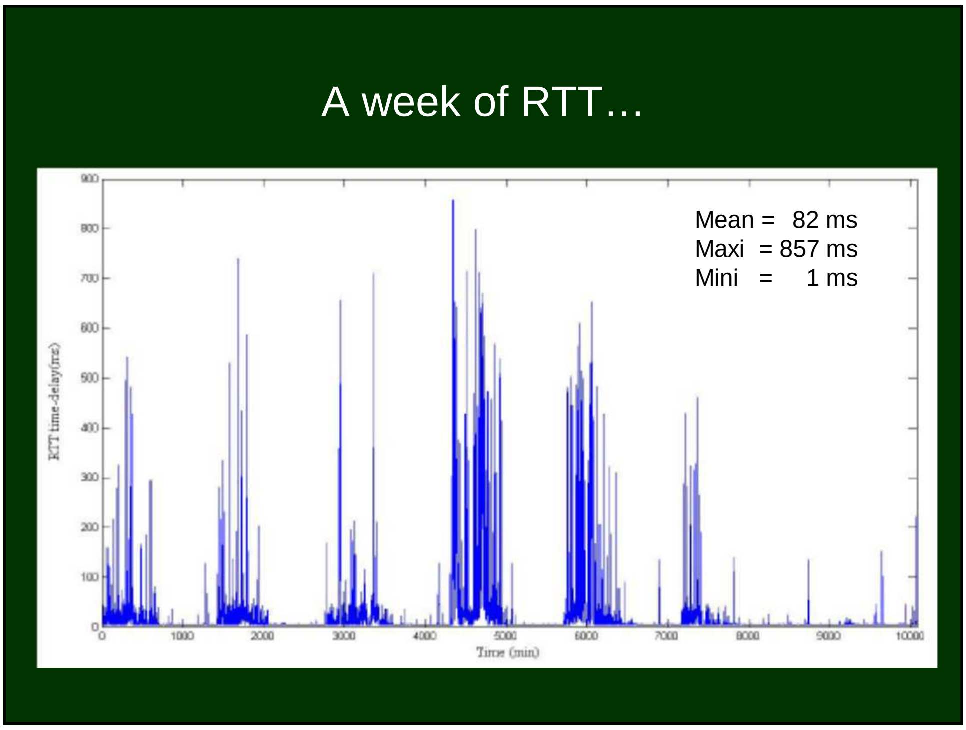


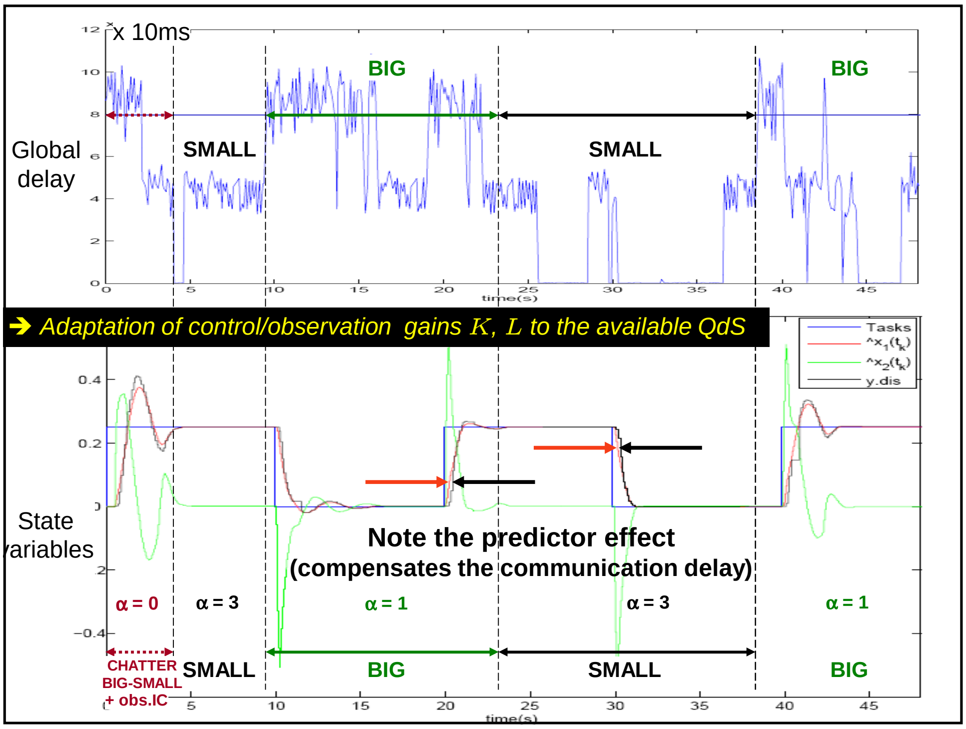
Another statement of the same problem...



- Influence of the *maximum* sampling period h_m
- Application of Fridman's criterion $\mathrm{d}h/\mathrm{d}t \le 1$


4th example : Networked Control Systems
Seuret et al. ACC 2006, Jiang et al. CDC'09, Kruszewski et al. IEEE CST 2011…
$\tau_1(t)$
Network
$h_1(t)$
Master
sampling delay
(40 km)
$h_2(t)$
Slave
sampling delay
remote observer + controller
$\tau_2(t)$
• Limited computation/power
$\dot{x}(t) = Ax(t) + Bu(t - \delta_1(t)),$
$y(t) = Cx(t).$
Internet:
+
Bluetooth:
Nighttime
vs
Daytime
+
Sampling:
+
Packet loss
$h_i + \tau_i = \delta_i$
homogeneous delay formulation


A week of RTT…
Mean = 82 ms
Maxi = 857 ms
Mini = 1 ms
RTT time-delay(ms)
Time (min)

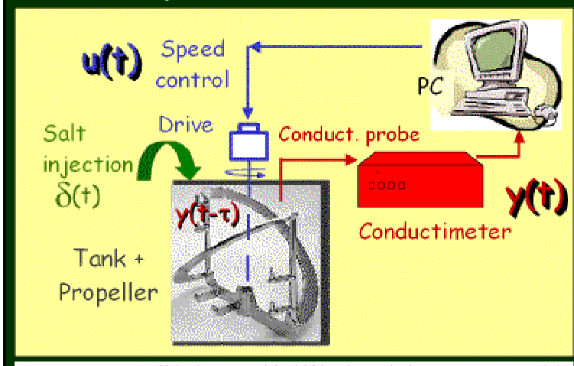


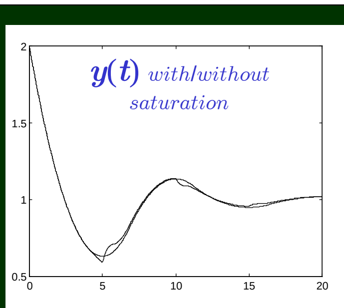
Goodies

Mixing tank system (with total recycle)

- Control = delay = k(motor speed)$^{-1}$
- Non flat system

$$T(u(t))\dot{y}(t) = y(t - h(u(t)) - y(t),$$





An arrangement of ideal zones with shifting boundaries as a way to model mixing processes in unsteady stirring conditions in agitated vessels

J.-Y. Dieulot[a,*], N. Petit[b], P. Rouchon[b], G. Delaplace[c]
Chemical Engineering Science 60 (2005) 5544–5554

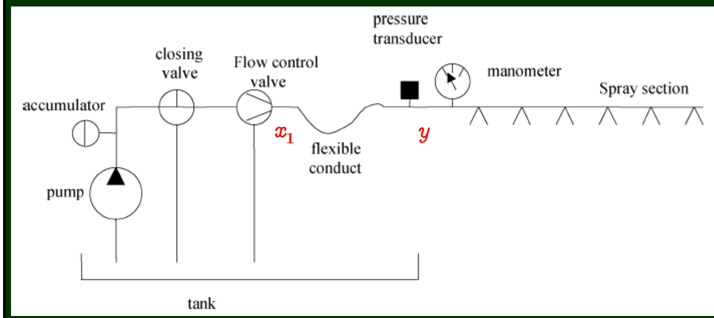
Dieulot, J. Y., & Richard, J. P. (2001). Tracking control of a nonlinear system with input-dependent delay. In *40th IEEE CDC'01 (Conference on decision and control)*, Orlando, FL, December 2001.

Goodies (2)

Design of a Pressure Control System With Dead Band and Time Delay

Jan Anthonis, Alexandre Seuret, Jean-Pierre Richard, and Herman Ramon

IEEE TRANSACTIONS ON CONTROL SYSTEMS TECHNOLOGY, VOL. 15, NO. 6, NOVEMBER 2007

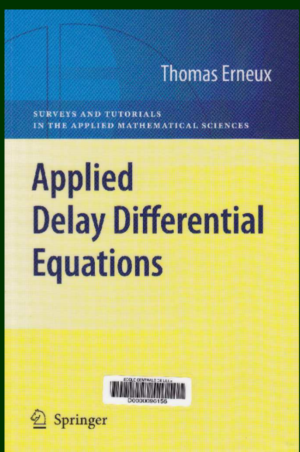


→ deadzone (dry friction on angle x_1)
\+ delay (measured pressure y)
\+ sign functions (control)

Pressure y ruled by the angle x_1

$$\sqrt{y} = \frac{\alpha x_1(t-h)+\beta}{x_1(t-h)+\gamma}, \quad 0 \le h_{\min} \le h \le h_{\max}.$$

Delays : various other examples



250 references taken from

- life sciences
- physics
- technology
- chemistry
- economics

OK, that's enough examples,

let's go to general questions

Contents

Distinctive features of TDS?

Illustrative examples
- 1rst example (remote crtl.) → basic notions (stability, state, inf. dim.)
- 2nd example : variable delay → counter-example
- 3rd example : sampling → delay for modelling
- 4th example : Networked Control System (master-slave)

Cauchy's problem for TDS
- Notion of solution
- Lipschitz-type condition
- And if the delay can vanish…

Stability and Lyapunov
- the LTI case
- 1rst Lyapunov (small states)
- small delays
- 2nd Lyapunov

Some words about identification

~~Mathematics..~~)
(just a little bit)

Cauchy's problem

(existence and unicity of solution for a TDS)

- Notion of solution
- Lipschitz-type condition
- And if the delay can vanish…

Cauchy pb. (1/4)

2.3 Notion of solution

System (S) : $\dot{x}(t) = f(t, x(t), x(t-\tau(t)))$

with $x(t) \in \mathbb{R}^n$, and $0 \leq \tau(t) \leq \tau$

Let $\varphi : [-\tau, 0] \longrightarrow \mathbb{R}^n$ be an arbitrary map.

Definition: *A map $x(t) : [t_0 - \tau, t_0 + b) \longrightarrow \mathbb{R}^n$ s.t.*
1) $x(t_0 + s) = \varphi(s)$, for all s in $[-\tau, 0]$;
2) x is continuous over $[t_0, t_0 + b)$;
3) x satisfies (S) over $[t_0, t_0 + b)$ ($\dot{x}$ right-hand, Dini)
is called a solution of (S) with initial value φ at t_0.
If only one map satisfies these 3 points, then the solution is unique.

Remark: There is a weaker notion of solution, where
2)$\longrightarrow$ x *absolutely continuous* function over $[t_0, t_0 + b)$
3)$\longrightarrow$ x satisfies (S) *almost everywhere* on $[t_0, t_0 + b)$

Cauchy pb. 2/4

2.4 Existence and uniqueness of solutions

For system (S) with $0 < \delta \leq \tau(t) \leq \tau_m$:

$$\dot{x}(t) = f(t, x(t), x(t-\tau(t))).$$

Consequence of the step method:
Given a continuous map $\varphi \in \mathcal{C}$, if the ODE

$$\dot{x}(t) = f_\varphi(t, x(t)) \equiv f(t, x(t), \varphi(t-\tau(t)))$$

has a (unique) solution, then there exists a (unique) solution of (S) with initial condition φ

From there, using classical Cauchy-Lipschitz cond^{tions}:
→Conditions of existence and uniqueness (I):

If f is a continuous map and satisfies a local Lipschitz condition in x,

$$\|f(t, x_2, y) - f(t, x_1, y)\| \leq K \|x_2 - x_1\|,$$

then for any initial condition $\varphi \in \mathcal{C}$, (S) has a unique solution, depending continuously on f and φ.

Cauchy pb. 3/4

If the delay can become zero, $0 \leq \tau(t) \leq \tau_m$ the step method does not apply anymore

⇒ need of a general framework: FDEs [Myshkis 49]

$$(RFDE): \quad \dot{x}(t) = F_R(t, x_t) \qquad \text{(retarded type)}$$

Conditions of existence and uniqueness (II):

If F_R is a continuous map with local-Lipschitz cond. in its second (functional) argument, i.e.

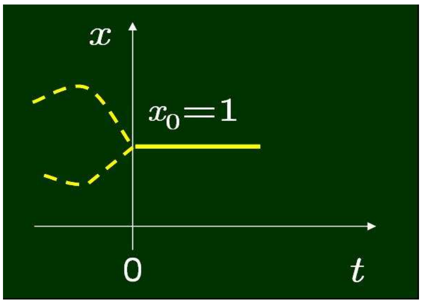
$$\|F_R(t, \varphi_2) - F_R(t, \varphi_1)\| \leq K \|\varphi_2 - \varphi_1\|_{\mathcal{C}}, \ldots$$

then for any initial condition $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, (RFDE) has a unique solution, depending continuously on F_R and φ.

This condition is also necessary for previous system (S) (discrete, bounded, nonzero delay)

Cauchy pb. 4/4

Remark



Even if unicity holds, different solutions may coincide after a finite time. For instance:

$$\dot{x}(t) = -x(t-\tau)[1 - x(t)],$$

$$x(t,\varphi) = 1 \quad (\forall t \geq 0)$$

for any $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{R})$ such that $\varphi(0) = 1$.

(non-unicity of the trajectory reversion)

Contents

Distinctive features of TDS?

Stability

- the LTI case
- 1[rst] Lyapunov (small states)
- small delays
- 2[nd] Lyapunov

Stability : the LTI case

Theorem:

A linear time-invariant system (thus,with a constant delay):

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h),$$

is globally asymptotically stable iff all its characteristic roots:

$$\det(sI - A_0 - e^{-hs}A_1) = \ldots 0$$

are in the strict left half plane .

Exemple 1. *Considérons l'équation* $\dot{x}(t) = -x(t-1)$. *Son équation caractéristique est* $s + e^{-s} = 0$, *dont les solutions* $s = \alpha \pm j\beta$ *sont en nombre infini. Le système n'est donc pas dégénéré. Ici,* $s = -0.318 \pm 1.337j$ *est une estimation de la paire de racines de plus grande partie réelle : il y a donc stabilité asymptotique*[2]. *Par contre, le cas suivant est dégénéré et instable :*

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} x(t-h),$$

$$\det(sI - A_0 - e^{-hs}A_1) = \ldots\ s(s^2-1).$$

Stability : 1rst method of Lyapunov

« approximation of the small deviations »

$$\dot{x}(t) = \sum_{i=0}^{k} A_i x(t - h_i) + q(t, x_t) \qquad (5.8)$$

$$q(t, x_t) = q(t, x(t), x(t - \tau_1(t)), ..., x(t - \tau_k(t)),$$
$$h_0 = 0,\ h_i = \text{constantes},\ \tau_j(t) \in [0, \tau_i] \text{ continues},$$
$$\|u_i\| \le \varepsilon \Rightarrow \|q(t, u_0, ..., u_k)\| \le \beta_\varepsilon(\|u_0\| + ... + \|u_k\|),$$

avec β_ε = constante pour ε donné, β_ε uniformément décroissante vers 0 quand $\varepsilon \to 0$. L'approximation au premier ordre est définie par :

$$\dot{z}(t) = \sum_{i=0}^{k} A_i z(t - h_i). \qquad (5.9)$$

If the linearized system (5.9) is asymptotically stable, then $z = 0$ is asymptotically stable for (5.8).
If (5.9) has at least one characteristic root with positive real part, then $z = 0$ is unstable for (5.8).

Stability: LTI with small delays

« approximation of small delays »

$$\dot{z}(t) = \sum_{i=0}^{k} A_i z(t - h_i). \qquad (5.9)$$

$$\dot{x}(t) = \sum_{i=0}^{k} A_i x(t - h_i) + q(t, x_t) \qquad (5.8)$$

The previous result (small variation approximation) can be usefuly completed with a *small delay approximation*, which is a qualitative result obtained from the continuity of the characteristic roots of (5.9) w.r.t. the delays h_i.

$$A = \sum_{i=0}^{k} A_i$$

Theorem:
If A is Hurwitz (resp., unstable), then for sufficiently small values of the delays h_i, $z = 0$ is asymptotically stable (resp., unstable) for (5.9) and, thus, for (5.8).

If A has one zero eigenvalue and all the other ones have negative real parts, then for sufficiently small values of the delays h_i, $z = 0$ is stable (resp., unstable) for (5.9) and, thus, for (5.8).

Stability: LTI with small, single delay

quantification of a « small » admissible delay

$$\frac{dz(t)}{dt} = A_0 z(t) + A_1 z(t-h), \qquad (5.10)$$

qui, pour un retard nul, devient :

Sufficient condition

$$\frac{dz(t)}{dt} = (A_0 + A_1)\, z(t). \qquad (5.11)$$

Théorème 6. [40] *Si le système à retard nul (5.11) est asymptotiquement stable et si P est la matrice solution de l'équation de Liapounov (où Q est une matrice réelle définie positive [117]) :*

$$(A_0 + A_1)^T P + P(A_0 + A_1) = -Q^T Q, \qquad (5.12)$$

alors (5.10) est asymptotiquement stable pour tout retard $h \in [0, h_{\max}]$:

$$h_{\max} = \frac{1}{2}\left[\lambda_{\max}(B^T B)\right]^{-\frac{1}{2}}, \quad avec\ B = Q^{-T} A_1^T P (A_0 + A_1) Q^{-1}. \qquad (5.13)$$

Lyapunov for TDS

Another result by V.B Kolmanovskii & A.D. Myshkis (1999)

$$\dot{x}(t) = Ax(t-h)$$

(also in nonlinear $\dot{x}(t) = f(t, x(t-h))$)

« dissipative systems »

$V(x_t) = ||x(t)||$ (some norm)

$||A||$ = associated matrix norm,

$\gamma(A)$ = logarithmic norm ("measure").

$\gamma(A) < -h||A||^2 \Longrightarrow$ expon. stable, $e^{-\omega t}$

ω : solution of $\omega = -\gamma(A) - h||A||^2 e^{2\omega h}$

Lyapunov's direct method for TDS

ODE :

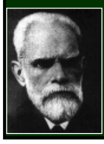
$\dot{x}(t) = -ax(t)$ ➡ $V(x(t)) = x^2(t) > 0$, $\dot{V}(x(t)) = -2\ ax^2(t) < 0 \ldots$ etc.

FDE :

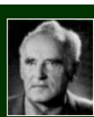
$\dot{x}(t) = -ax(t) - bx(t-h)$

$V(x(t)) = x^2(t)$ (« usual » quadratic)

$\dot{V}(x(t)) = -2\ [ax^2(t) + \boxed{bx(t)x(t-h)}] \leq \ldots ?$ *cross terms*



➔ need of delay-dedicated methods :



1) Lyapunov-Razumikhin functions (not here)

2) Lyapunov-Krasovskii functionals

Lyapunov for TDS

an illustration of the Lyapunov-Krasovskii approach

$\dot{x}(t) = -ax(t) - bx(t-h)$

$V(x_t) = x^2(t) + |b| \boxed{\int_{-h}^{0} x^2(t+s)ds}$ (quad + integral)

$$\dot{V}(x_t) = -2x(t)[ax(t) + bx(t-h)] + |b|[x^2(t) - x^2(t-h)]$$

$$\leq -2(a - |b|)x^2(t) \quad \ldots\ \dot{V}(x_t) < 0 \text{ if } |b| < a$$

NB: LK-functionals were used in the above NCS/sampled data proofs
(under a much more general form)

A bit more general LKF…

$(S) \quad \dot{x}(t) = Ax(t) + Bx(t-\tau)$, avec $x(t) \in \mathbb{R}^n$.

Fonctionnelle : $\mathcal{V}(\varphi) = \varphi^T(0)P\varphi(0) + \int_{-\tau}^{0} \varphi^T(s)S\varphi(s)\,\mathrm{d}s$
avec $P,\ S \succ 0$.

$$\Rightarrow \boxed{\dot{\mathcal{V}}(x_t) = y^T(t)Qy(t)}$$

avec $Q = \begin{bmatrix} A^TP + PA + S & PB \\ B^TP & -S \end{bmatrix}$ et $y(t) = \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}$

$\Rightarrow$ Stabilité asymptotique i.d.r. si $Q \prec 0$ (LMI)

and a way to delay-dependent stabilty…

Hyp. : (S) asymp. stable pour $\tau = 0 \Rightarrow \mathcal{A} = A + B$ Hurwitz

Problème

Chercher une borne τ^* t.q. stab. asympt. $\forall \tau \leq \tau^*$.

Idée

Transformation du modèle à l'aide de la formule de Leibniz-Newton :

$$x(t) - x(t-\tau) = \int_{-\tau}^{0} \dot{x}(t+s)\,\mathrm{d}s$$

(S)$\Rightarrow$

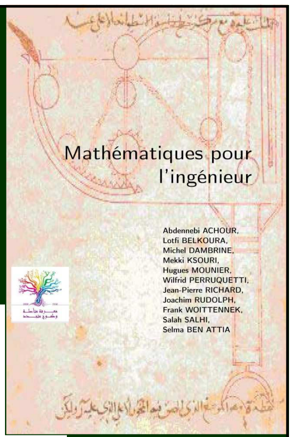
$$\dot{x}(t) = (A+B)x(t) - B\int_{t-\tau}^{t} (Ax(s) + Bx(s-\tau))\,\mathrm{d}s$$

+ many more general LKFs

see the textbook *Mathématiques pour l'Ingénieur*

ISBN : 978-9973-0-0852-7 (Tunisie) *385 pages,* 2009

pdf also available at http://hal.inria.fr/hal-00519555_v1/



$$\dot{x}(t) = \sum_{i=1}^{m} A_i x(t - h_i). \qquad (6.47)$$

$$A = \sum_{i=1}^{m} A_i, \quad A_{ij} = A_i A_j, \quad h_{ij} = h_i + h_j, \quad h = \sum_{i=1}^{m} h_i. \qquad (6.48)$$

Théorème 6.5.8. *Le système (6.47) est asymptotiquement stable si, pour deux matrices symétriques et définies positives R, Q, il existe une matrice définie positive P solution de l'équation de Riccati :*

$$A^T P + PA + mRh + P \sum_{i,j=1}^{m} h_i A_{ij} R^{-1} A_{ij}^T P = -Q. \qquad (6.49)$$

Démonstration : on choisit la fonctionnelle $V = V_1 + V_2$, $V_1 = x^T(t)Px(t)$, $V_2 = \sum_{i,j=1}^{m} \int_{h_j}^{h_{ij}} ds \int_{t-s}^{t} x^T(\tau)Rx(\tau)d\tau$, conduisant à $\dot{V} = -x^T(t)Qx(t) - \sum_{i,j=1}^{m} \int_{t-h_j}^{t-h_{ij}} [Rx(\theta) + A_{ij}^T Px(t)]R^{-1}[Rx(\theta) + A_{ij}^T Px(t)]^T d\theta$. ■

Contents

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- 1rst example (remote crtl.) → basic notions (stability, state, inf. dim.)
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- 4th example : **Networked Control System** (master-slave)

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- And if the delay can vanish…

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- small delays
- 2nd Lyapunov

… Few last words about identification

Some words on our current research in
identification/estimation

Non-A : Non-Asymptotic estimation for on-line systems

http://www.inria.fr/equipes/non-a

a foreword on identifiability

INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control 2003; **13**:857–872 (DOI: 10.1002/rnc.850)

Adaptive identification of linear time-delay systems

Y. Orlov[1,*,†], L. Belkoura[2,‡], J.P. Richard[3,§] and M. Dambrine[3,¶]

$$\dot{x}(t) = \Sigma_{i=0}^{r}[A_i x(t-\tau_i) + B_i u(t-\tau_i)], \quad (1)$$

$$\dot{x}(t) = A(\lambda)x(t) + B(\lambda)u(t), \quad (6)$$

$$y(t) = C(\lambda)x(t) \quad (7)$$

over the ring $\mathbf{R}[\lambda]$ of polynomials in a vector variable $\lambda = (\lambda_1, ..., \lambda_k)^T$

In matrix terms system (1) is weakly controllable iff for some $z \in \mathbf{C}^k$.

$$rank\ \left[B(z) \mid A(z)B(z) \mid \ldots \mid A^{n-1}(z)B(z)\right] = n \quad (9)$$

Definition 1 *System (1) is said to be identifiable if there exists a control input $u(t)$ such that the identity $x(t) \equiv \hat{x}(t)$ results in*

$$r = \hat{r},\ \tau_i = \hat{\tau}_i,\ A_i = \hat{A}_i,\ B_i = \hat{B}_i\ for\ i = 0, \ldots, r,$$

regardless of a choice of the initial functions $\varphi(\theta), \hat{\varphi}(\theta)$. In that case the identifiability is said to be enforced by the control input $u(t)$.

Theorem 1 *The time-delay system (1) is identifiable if and only if it is weakly controllable. Moreover, if (1) is weakly controllable then the identifiability can be enforced by any sufficiently nonsmooth control input $u(t)$.*

see also
by the same authors:

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 47, NO. 8, AUGUST 2002
On Identifiability of Linear Time-Delay Systems

… and, more general (convolutions):

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Available online at www.sciencedirect.com
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automatica
Automatica 41 (2005) 505–512
www.elsevier.com/locate/automatica

Identifiabilty of systems described by convolution equations☆

Lotfi Belkoura*

Throughout this paper, we assume stable linear and time invariant systems with input $u(.) \in \mathbb{R}^q$ and output $y(.) \in \mathbb{R}^p$. The models considered are those that can be described by one the following convolution equations:

$$P * y = Q * u, \quad (1)$$

$$P * z = u, \quad y = Q * z. \quad (2)$$

Identification results: up to now, linear systems

➔ Adaptive (continuous) techniques

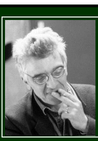
- Diop, Kolmanovskii, Moraal, vanNieuwstadt – Control Eng.Pract. 9, 2001
  "Preserving stability/performance when facing an unknown time delay."
- Orlov, Dambrine, Belkoura, Richard - IJNRC 13, 2003
  (see above)
- Gomez, Orlov, Kolmanovskii – Automatica 43(12) 2007
  "On-line identification of SISO linear time-invariant delay systems from output measurements"

➔ Nonsmooth techniques (VSS)

- Drakunov, Perruquetti, Richard, Belkoura, - Ann. Reviews in Control, 30(2) 2006
  "Delay identification in time-delay systems using variable structure observers"

➔ Algebraic techniques (distributions)

- Belkoura, Richard, Fliess - Automatica 45(5) 2009
  "Parameters estimation of systems with delayed and structured entries"



The idea of algebraic estimation

(Fliess, Sira-Ramirez ESAIM COCV 2003)

Basic example, no-delay case

$$\dot{y}(t) = ay(t) + u(t) + \gamma$$

a : unknown parameter;
γ : constant perturbation

$$s\hat{y}(s) = a\hat{y}(s) + \hat{u}(s) + y_0 + \frac{\gamma}{s}$$ (y_0 : initial condition)

- elimination of γ :

$$\frac{d}{ds}\left[s\left\{s\hat{y}(s) = a\hat{y}(s) + \hat{u}(s) + y_0 + \frac{\gamma}{s}\right\}\right]$$

$$\Rightarrow 2s\hat{y}(s)+s^2\hat{y}'(s) = a\left(s\hat{y}'(s) + \hat{y}(s)\right)+s\hat{u}'(s)+\hat{u}(s)+y_0$$

Algebraic estim., Ctn'd

- estimation of a: $(y_0 = 0)$

$$s^{-\nu}[2s\hat{y}(s) + s^2\hat{y}'(s) = a(s\hat{y}'(s) + \hat{y}(s)) + s\hat{u}'(s) + \hat{u}(s)]; \quad \nu > 0$$

$$y'(s) = \frac{dy(s)}{ds} = \mathcal{L}(-ty(t))$$

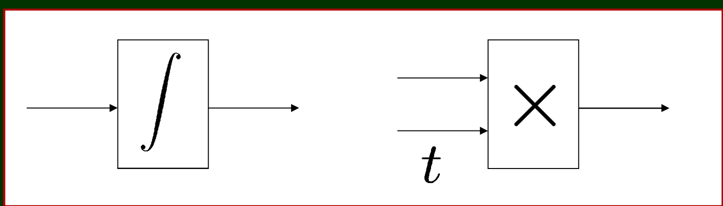
$$a = \frac{2\int_0^t d\lambda \int_0^\lambda y(\tau)d\tau - \int_0^t \tau y(\tau)d\tau + \int_0^t d\lambda \int_0^\lambda \tau u(\tau)d\tau - \int_0^t d\lambda \int_0^\lambda d\sigma \int_0^\sigma u(\tau)d\tau}{\int_0^t d\lambda \int_0^\lambda d\sigma \int_0^\sigma y(\tau)d\tau - \int_0^t d\lambda \int_0^\lambda \tau y(\tau)d\tau}$$

$\nu = 3$

- t may be very small → fast estimation
- number ν of integrations → averaging role

Algebraic estim., Ctn'd

$$a = \frac{2\int_0^t d\lambda \int_0^\lambda y(\tau)d\tau - \int_0^t \tau y(\tau)d\tau + \int_0^t d\lambda \int_0^\lambda \tau u(\tau)d\tau - \int_0^t d\lambda \int_0^\lambda d\sigma \int_0^\sigma u(\tau)d\tau}{\int_0^t d\lambda \int_0^\lambda d\sigma \int_0^\sigma y(\tau)d\tau - \int_0^t d\lambda \int_0^\lambda \tau y(\tau)d\tau}$$

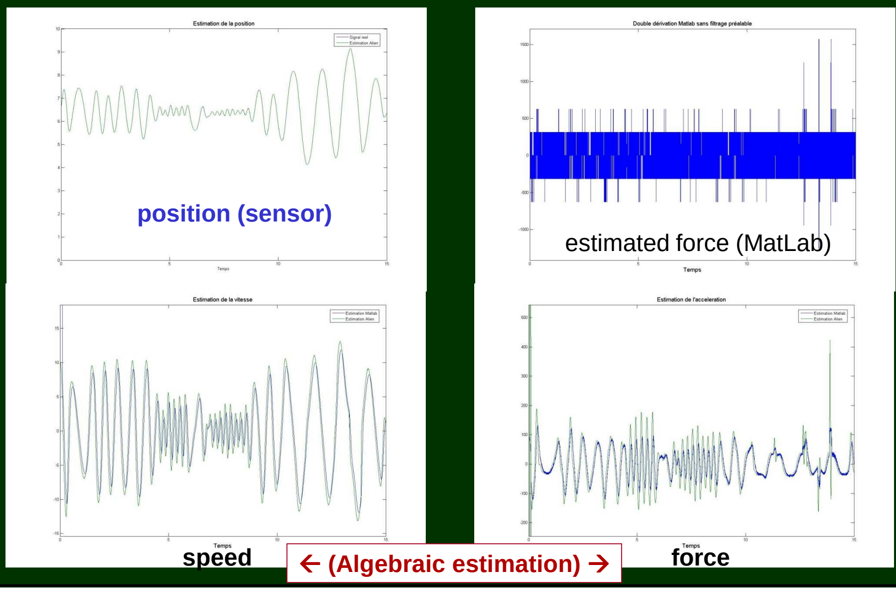


- parameters or states obtained via iterative integrations (or, more generally, low-pass filters)
- noise = fast fluctuations

Algebraic estim., Ctn'd

Same approach works for derivative estimation

[Mboup, Join, Fliess, Numer. Algor. 2009]
see also [Riachy, Mboup,Richard, J. of Comput.& Applied Math., to appear]

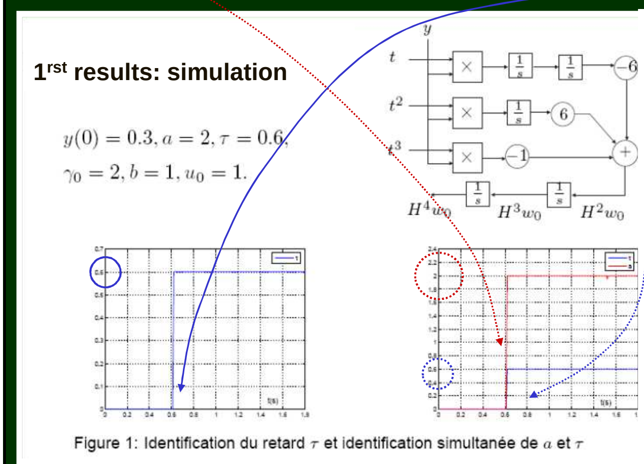


Algebraic estim., Ctn'd

... and it works for delay estimation

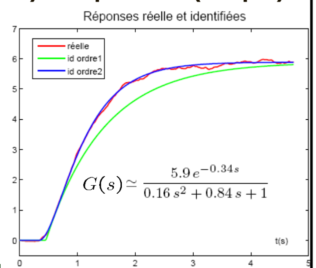
[Belkoura, Richard, Fliess, Automatica 2009]

$$\dot{y}(t) + ay(t) = y(0)\delta + \gamma_0 H + bu(t-\tau)$$ (basic example)



$$\tau = \frac{H^k\left(t^3 y^{(2)} + a t^3 y^{(1)}\right)}{H^k\left(t^2 y^{(2)} + a t^2 y^{(1)}\right)}, \quad t > \tau.$$

2) real process (simple)



Algebraic estim., Ctn'd

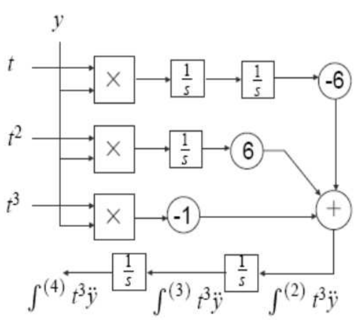
The integrations are performed using the integration by part formula, avoiding any derivative in the algorithm.

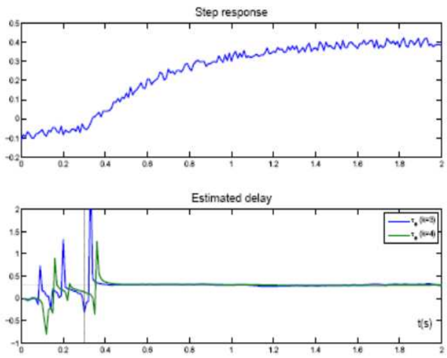
$$\text{Ex:} \quad \int^{(k)} t^3 \ddot{y} = -6 \int^{(k)} ty + 6 \int^{(k-1)} t^2 y - \int^{(k-2)} t^3 y. \qquad (6)$$

Remark: Filters may be used instead of integrals.

$$\tau = \frac{H^k\left(t^3 y^{(2)} + a t^3 y^{(1)}\right)}{H^k\left(t^2 y^{(2)} + a t^2 y^{(1)}\right)}, \qquad t > \tau.$$

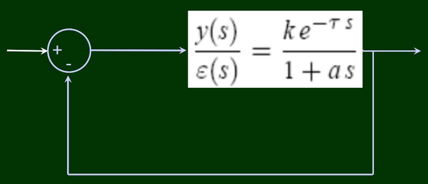
Partial realization scheme and simulation results.

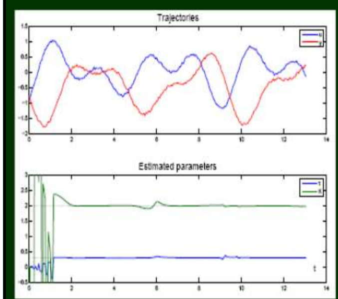




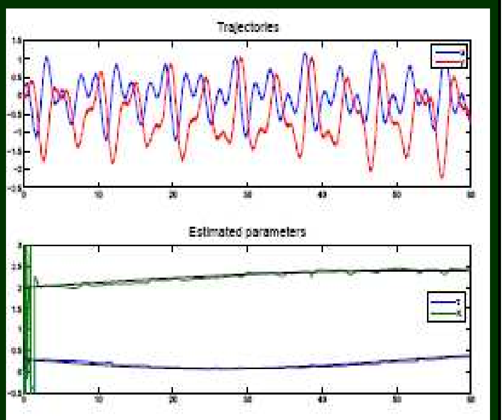
Algebraic estim., Ctn'd

... and may work for closed loop, variable delay estimation





constant parameters



slowly varying parameters

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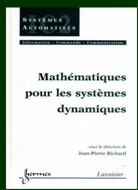

…or →