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T. Kailath

State-Space Modelling: State-Space Transformation

Recent developments in automatic control demonstrate the basic importance of the "modelling" phase. This concept is rather wide, since it includes both identification of the physical devices (with measurement and data processing) and adaptation of the mathematical formalisms (in relation to the nature of the system and the aims of the study). Among these possible formalisms, state-space representation appears to be particularly convenient for many kinds of studies, such as optimal control and stability or sensitivity analysis. The advantages of such a representation are numerous; for example,

(a) it is applicable to both differential and recurrent systems,

(b) it is available for nonlinear or time-varying processes and

(c) it allows the use of matrix algebra and vector notation.

1. Model Adaptation

The system is described by usual state-space equations such as

$$\left.\begin{aligned}\delta x &= Ax + Bu\\ y &= Cx + Du\end{aligned}\right\} \tag{1}$$

where δ is the derivative ($\delta x = \mathrm{d}x/\mathrm{d}t$) or iteration ($\delta x(t) = x(t+T)$) operator, x is the state vector, u is the input, y is the output and A, B, C, D are matrices (constant or not). A is called the state matrix, B is the input matrix and C is the output matrix.

However, a fundamental property of the state-space representation lies in its nonuniqueness: changing the basis in which vectors are described leads to other state-space equations which can be more helpful for analysis. Indeed, if one considers a matrix $P(t)$ defining a Lyapunov transformation

$$\left.\begin{array}{l}P(t)\text{ is constant, or depends on time } t\\ P(t)\text{ has an inverse } P^{-1}(t)\text{ for any } t\\ P(t),\ P^{-1}(t)\text{ and } \delta P(t)\text{ and continuous and}\\ \quad\text{bounded for any } t\end{array}\right\} \tag{2}$$

then the new state vector

$$\bar{x} = P(t)x$$

obeys the equations

$$\left.\begin{aligned}\delta\bar{x} &= \bar{A}\bar{x} + \bar{B}u\\ y &= \bar{C}\bar{x} + Du\end{aligned}\right\} \tag{3}$$

The expressions of $\bar{A}$ and $\bar{B}$ depend on the nature of the operator δ. If δ is the derivative, then

$$\left.\begin{aligned}\bar{A} &= PAP^{-1} + \delta P P^{-1}\\ \bar{B} &= PB\end{aligned}\right\} \tag{4}$$

and if δ is the iteration operator, then

$$\left.\begin{aligned}\bar{A} &= \delta PAP^{-1}\\ \bar{B} &= \delta PB\end{aligned}\right\} \tag{5}$$

The matrices A and $\bar{A}$ are then said to be topologically equivalent.

This article deals with a particular case of Lyapunov transformations, since we consider the case of constant matrices P, starting from a (supposed known) initial state-space representation of the same type as Eqns. (1). In this case, several advantages arise.

(a) The representations in Eqns. (4) and (5) are equal for both the derivative and iteration, and $\bar{A}$ is then similar to A (Eqns. (1) and (3) are said to be equivalent).

(b) The stability properties of A and $\bar{A}$ are identical (as for any Lyapunov transformation).

(c) The controllability and observability properties (defined for A, B, C, D constant) are the same for Eqns. (1) and (3).

(d) The nature of the dynamical behavior of the model of Eqns. (1) is conserved: observing $x(t)$ or $Px(t)$ gives the same results, which would not be the case if P was, for example, a periodic matrix $P(t)$, introducing or suppressing oscillations in the observation of $P(t)x(t)$.

(e) Lastly, as we shall see, a constant change-of-basis matrix P allows the use of numerous results of algebra, such as matrix similarity or characteristic polynomial calculations.

Thus, what we call model adaptation is the choice of a modified model, Eqns. (3) (and then of a change-of-basis P) in relation to the possibility of analysis linked to each state-space representation. In the linear time-invariant case (A, B, C, D constant matrices), this adaptation is of great interest for the convergence of numerical algorithms, for simulation or for demonstrating input–output properties. However, in the general case of a nonconstant matrix A, all the analysis and synthesis problems are closely linked to the state basis choice. A good example of this is provided by the stability problem: since A is not constant, it is generally impossible to obtain necessary and sufficient stability conditions by analytic methods. The conditions obtained being generally only sufficient, they depend on the state basis choice and therefore on the model adaptation.

The choice of the changing matrix P has then to be directed so as to obtain a "well-conditioned" matrix $\bar{A}$, which will provide an efficient performance analysis of the system properties. With this aim in mind, four interconnected goals have to be considered: block matrix partitioning, reduction of the number of nonzero elements, reduction of the number of nonconstant coefficients and the nature of the diagonal terms.

(*a*) *Block matrix partitioning.* Partitioning may be used simply for numerical analysis (Schur formula, Kronecker products, matrix polynomials) or with a view to significant decomposition into subsystems: this is the case in studies of multivariable, hierarchical or decentralized control, of singularly perturbed systems or of stability based on vector norms.

(*b*) *Reduction of the number of non-zero elements.* The goal is to simplify the matrix expression and calculations, and leads to the notion of hollow matrix (presenting a large number of zero elements) or dense matrix (presenting a small number of zero elements). It is also connected with the partitioning problem and with block triangular or diagonal decomposition, or with the reduction of the number of significant terms (canonical forms, symmetric or skew-symmetric forms, and so on).

(*c*) *Reduction of the number of nonconstant coefficients.* This aims at approaching the easier linear time-invariant case, allows a more direct application of some stability criteria and leads to more efficient simulation diagrams. Starting from an initial model as in Eqns. (1), this can be achieved by choosing an appropriate state basis, by aggregation and/or overvaluing techniques or by other structural reduction methods.

(*d*) *Nature of the diagonal terms.* The matrix diagonal has also to be taken into account: if the system is of differential type, it is preferable for the diagonal terms to present negative real parts and to be of modulus less than one in the discrete case (mainly for stability reasons). In all cases, the notion of diagonal dominance is useful (the absolute value of each diagonal term is then greater than the sum of the absolute values of the terms of the corresponding row or column); diagonal dominance is used in eigenvalue location problems and is related to the questions of partitioning and of obtaining a hollow matrix.

Several authors have presented canonical representations of linear or nonlinear systems. The positions of the zero coefficients and of the nonconstant components define a "matrix form" whose terms have to be computable starting from a knowledge of any other noncanonical representation. This is the case for constant matrices of diagonal, Jordan, companion, Hankel, Toeplitz or Jacobi forms (see Rosenbrock and Storey 1970, Gantmacher 1959). For linear nonautonomous multivariable systems, an important list of references on Kronecker, Hermite, Bosgra, Hessenberg and Brunovsky forms is given in Hinrichsen (1985), associated with parametrization problems.

Some of these forms have been generalized to nonlinear systems, as is the case for companion, diagonal or Jordan-type matrices. Because of the constraints imposed by nonconstant coefficients, these generalizations often present one or several nonzero rows or columns. In the case of modal forms (diagonal or Jordan), the corresponding matrices are of "arrow" type, and these will be widely featured in this article.

Even when the required matrix is not in canonical form, the practical problem is often stated as follows.

Problem 1

(a) Let (A, B, C) be any initial model of the system.

(b) Let a matrix form $\bar{A}$ be chosen whose conditioning is adapted to the study to be conducted. The coefficients $(\bar{a}_{ij})$ of $\bar{A}$ are not yet all determined (some being imposed, others not), but their location in $\bar{A}$ is assigned.

(c) What are the conditions on the $(\bar{a}_{ij})$ which guarantee the existence of a matrix $\bar{A}$ similar to A?

(d) What is the corresponding change of basis from A to $\bar{A}$?

Problem 2

(a) Let (A, B, C) be any initial model of the system.

(b) Let a regular constant matrix P be chosen, of the same dimensions as A.

(c) $\bar{A} = PAP^{-1}$ represents the same system.

(d) Is $\bar{A}$ convenient for the study to be conducted?

This second problem has solutions which are less useful than those of Problem 1 but easier to implement and can be solved in all cases. We shall see later that Problem 1 can be solved for large classes of systems by introducing the notion of a symbolic polynomial.

2. *Linear Systems*

In this section the matrices (A, B, C) are assumed to be constant. The usual canonical forms for the state matrix A are the Jordan form and the first natural canonical form. The Jordan form is composed of a diagonal of real or complex terms, and an upper-diagonal of 1 or 0:

$$A_J = \left[\begin{array}{c|c|cc|ccc} * & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & * & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & * & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & * & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & * \end{array}\right] \tag{6}$$

The diagonal elements of A are necessarily the eigenvalues of A. If these eigenvalues are all different, there is no upper diagonal term 1. When the eigenvalues are complex, of type $\alpha \pm j\beta$, it is possible to make rotation blocks appear in the diagonal, of type

$$R = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

The first natural canonical form is composed of a block diagonal of companion matrices. Each companion matrix is associated with one of the invariant polynomials of A. The product of these invariant polynomials is the characteristic polynomial of A:

$$A_{\mathrm{FNC}} = \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & * & 0 & 0 & 0 \\ 1 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 1 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 1 & * & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \end{array}\right] \tag{7}$$

It is important to notice that two matrices A_1 and A_2 are similar if and only if they have the same invariant polynomials. However, possession of the same characteristic polynomial is only a necessary condition of similarity.

When one considers the canonical forms of the nonautonomous system (hence the pair (A, B)), the natural canonical form presents several supplementary nonzero terms. For example, the Kronecker canonical form of a pair (A, B) can be

$$A_K = \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & * & 0 & 0 & * \\ 1 & 0 & 0 & * & 0 & 0 & * \\ 0 & 1 & 0 & * & 0 & 0 & * \\ 0 & 0 & 1 & * & 0 & 0 & * \\ \hline 0 & 0 & 0 & * & 0 & 0 & * \\ 0 & 0 & 0 & * & 1 & 0 & * \\ 0 & 0 & 0 & * & 0 & 1 & * \end{array}\right], \quad B_K = \left[\begin{array}{ccc} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{8}$$

The matrix A_K is slightly denser than A_{FNC}, because B_K cannot be chosen in any form (the ith block of B_K must have its $(i-1)$ first columns equal to zero, and all its nonzero elements in its first row).

The matrices A_J, A_{FNC} or A_K also exist in their transposed forms, but the list given here is restricted to the models to be used in the subsequent sections. Among them, the Brunovski canonical form of a pair (A, B) is of type

$$A_B = \left[\begin{array}{cccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ * & * & * & * & * & * & * \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ * & * & * & * & * & * & * \end{array}\right], \quad B_B = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & * & * \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & * \end{array}\right] \tag{9}$$

3. *Classification of Nonlinear Representations*

In what follows, I_n denotes the identity $n \times n$ matrix, and $0(m, n)$ the zero $m \times n$ matrix.

3.1 *Definitions of Classes l(k) and c(k)*

The first class of nonlinear models considered is described by

$$\left.\begin{array}{l} \delta x = A(\cdot)x + B_0(\cdot)e \\ A(\cdot) = A_0 + B(\cdot)C \\ y = C_0(\cdot)x \\ C \text{ is of rank } k \\ A_0 \text{ is a constant } q \times q \text{ matrix} \end{array}\right\} \tag{10}$$

The state x is of order q; the input e and the output y are of order k, with $k \leqslant q$.

The matrices $B_0(\cdot)$ and $B(\cdot)$ are not constant, and depend on time t, on state x, command u or on any other parameters (denoted $(\cdot)$). However, it is necessary for Eqn. (10) to have a unique solution, and $B_0(\cdot)$ and $B(\cdot)$ are assumed to verify adequate properties. Autonomous behavior ($e \equiv 0$) is governed by the state matrix, Eqn. (11), which is said to be of type $c(k)$:

$$A(\cdot) = A_0 + B(\cdot)C \tag{11}$$

This class of systems (Eqns. (10)) is called systems with nonlinearities of rank k, of type $c(k)$, from the fact that the nonconstant terms of $A(\cdot)$ can be regrouped in only its k last columns; indeed, since C is of rank k, the k last components of the state vector x can be chosen equal to Cx, and then, correspondingly,

$$C = [0(k, q-k), I_k] \tag{12}$$

$$A(\cdot) = A_0 - [0(q, q-k), B(\cdot)] \tag{13}$$

When the matrix $B(\cdot)$ can be developed into the following special form, the system is of Lur'e–Postnikov type:

$$\left.\begin{array}{l} B(\cdot) = BF(\epsilon, t) = B_0(\cdot) \\ C_0(\cdot) = C \in \mathbb{R}^{k\times q} \text{ constant matrix} \\ B \in \mathbb{R}^{q\times k} \text{ constant matrix} \\ \epsilon = e - y \\ F(\epsilon, t) = \mathrm{diag}\{\phi_i(\epsilon, t)\} \in \mathbb{R}^{k\times k} \\ F(\epsilon, t) \text{ is continuous, bounded for every} \\ \quad \text{bounded } (\epsilon, t) \end{array}\right\} \tag{14}$$

The functions ϕ_i are the equivalent gains of the system. The class represented by Eqns. (14) constitutes the most generally employed nonlinear state-space model for physical processes. It can be interpreted as a linear system of the type of Eqns. (1) with nonlinear feedback control, defined by

$$u = F(\epsilon, t)\cdot\epsilon \tag{15}$$

In other cases it can be fruitful to consider regrouping the nonconstant terms of $A(\cdot)$ with regard not to its columns but to its rows. In an analogous way, the system is said to be of type $l(r)$ (with nonlinearities of rank r) if it presents the decomposition

$$\left.\begin{array}{l} \delta x = A(\cdot)x + B_0(\cdot)e \\ A(\cdot) = A_0 + BC(\cdot) \\ y = C_0(\cdot)x \\ B \text{ is assumed to be of rank } r \end{array}\right\} \tag{16}$$

The notation and existence hypothesis are analogous to Eqns. (10). Autonomous behavior is governed by the $l(r)$ state matrix

$$A(\cdot) = A_0 + BC(\cdot) \tag{17}$$

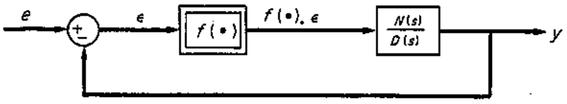


Figure 1
Example of a continuous system

The transposed matrix of $A(\cdot)$ is of type $c(r)$, and therefore the nonconstant terms of $A(\cdot)$ can be regrouped in its last r rows.

From these definitions, it follows that any system described by Eqns. (1) is of both types $c(k)$ and $l(r)$, with in the extreme case $k = r = q$ (if the q^2 components of $A(\cdot)$ are nonconstant). However, the smaller k or r is, the more efficient the proposed modelling method will be. It is often preferable to consider the decomposition (l or c) which leads to the smallest rank, namely, $l(r)$ if $r < k$, and $c(k)$ if $r > k$.

3.2 *Example*

Consider the continuous system defined in Fig. 1. The input e and output y are scalar. The transfer function is of qth order and is assumed not to be degenerate. In this system,

$$N(s) = \sum_{i=0}^{q-1} b_i s^i$$

$$D(s) = s^q + \sum_{i=0}^{q-1} a_i s^i$$

This system is of both types $l(1)$ and $c(1)$, since its equations are

$$\left.\begin{array}{l} \overset{\circ}{x} = A(\cdot)x + Bf(\cdot)e \\ A(\cdot) = A_0 - Bf(\cdot)C \\ y = Cx \\ f(\cdot) \in \mathbb{R}, \quad B \text{ and } C \text{ of rank } 1 \end{array}\right\} \tag{18}$$

with for example,

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & \cdots & \cdots & \cdots & -a_{q-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C^{\mathrm{T}} = \begin{bmatrix} b_0 \\ \vdots \\ \vdots \\ b_{q-1} \end{bmatrix} \tag{19}$$

Therefore, adequate changes of basis can bring Eqn. (18) into forms which regroup the nonconstant terms of $A(\cdot)$ into one row only or one column only: Eqns.

(18) and (19) directly lead to an $l(1)$-type matrix $A(\cdot)$. Choosing y as the last component of x brings $A(\cdot)$ into a $c(1)$ form.

3.3 Canonical Forms for c(k) and l(k) Systems

It is possible to bring any $c(k)$ matrix into a canonical form which is analogous to the Kronecker form A_K, Eqn. (8), but including nonconstant coefficients (asterisks in Eqn. (8) then become nonconstant terms). A possible proof is to choose the change-of-basis P by putting the pair (A_0^T, C^T) in its canonical Brunovski form (A_B, B_B) (as symbolized in Eqn. (9)); then $P^{-1^T}A(\cdot)P^T$ (with $A(\cdot)$ defined in Eqn. (11)) is of the desired type.

For $c(k)$ nonlinear systems, one may prefer a somewhat different arrangement of the nonconstant coefficients which reorganizes the nonconstant columns as the k last ones. The $l(k)$ systems can be considered in the same way; the results (as in Eqn. (8)) have then to be transposed (as in Eqn. (9)).

4. Invariant Polynomial for Nonlinear Systems

4.1 Definition of the Symbolic Polynomial for c(1) or l(1) Types

In this section, the matrices considered are of type $c(1)$ (Eqn. (20)) or $l(1)$ (Eqn. (21)):

$$A(\cdot) = A_0 + b(\cdot)c^T, \quad b(\cdot) = [b_i(\cdot)]_{i=1,\ldots,q} \tag{20}$$

$$A(\cdot) = A_0 + c^T b(\cdot) \tag{21}$$

Here c is a constant q vector, $b(\cdot)$ is a nonconstant q-vector.

DEFINITION 1. The symbolic polynomial of Eqn. (20) or (21) is defined by

$$p[\lambda, A(\cdot)] = \det[\lambda I_q - A(\cdot)] \tag{22}$$

This determinant is calculated as if $A(\cdot)$ was constant, and its expression can be developed into

$$p[\lambda, A(\cdot)] = p_0(\lambda) + \sum_{i=1}^{q} b_i(\cdot)p_i(\lambda) \tag{23}$$

where $p_i(\lambda)$ are constant-coefficient polynomials in λ.

Property. The symbolic polynomial is invariant through any constant change-of-basis P.

DEFINITION 2. The polynomial $p[\lambda, A(\cdot)]$ is said to be nonfactorizable iff it does not present any zero, independent of time, state and so on, that is,

$$\nexists \lambda_0 \in \mathbb{C}, \quad p[\lambda_0, A(\cdot)] \equiv 0, \quad \forall b_i(\cdot) \in \mathbb{R} \tag{24}$$

THEOREM 1. *Let two matrices $A(\cdot)$ and $\bar{A}(\cdot)$ be of the same type $c(1)$ (or $l(1)$), each satisfying one of the two conditions (a) $p[\lambda, A(\cdot)]$ or $p[\lambda, \bar{A}(\cdot)]$ is nonfactorizable, or (b) the pair (A_0^T, c) or $(\bar{A}_0^T, \bar{c})$ (or (A_0, c) or $(\bar{A}_0, \bar{c})$) is controllable, that is,*

$$\operatorname{rank}\{c, A_0^T c, \ldots, A_0^{T(q-1)}c\} = q \tag{25}$$

Then a necessary and sufficient condition for $A(\cdot)$ and $\bar{A}(\cdot)$ to be similar is that they have the same symbolic polynomial:

$$p[\lambda, A(\cdot)] \equiv p[\lambda, \bar{A}(\cdot)] \Leftrightarrow \exists P^{-1}, \quad \bar{A}(\cdot) = PA(\cdot)P^{-1}$$

REMARKS.

(a) According to Theorem 1, the symbolic polynomial appears as a practical tool for state-space adaptation. Indeed, it permits Problem 1 (Sect. 1) to be solved, by providing a necessary and sufficient test of similarity. Section 4.2 will show that it also allows the change-of-basis calculation.

(b) If (a) or (b) in Theorem 1 holds, $A(\cdot)$ and $\bar{A}(\cdot)$ are similar to simple nonconstant companion matrices.

(c) A major advantage of $p[\lambda, A(\cdot)]$ is that it is calculable by computer (see for example Richard 1984).

(d) In the classical case of a Lur'e–Postnikov-type system (Fig. 1), the expression of the symbolic polynomial is directly given by

$$p[\lambda, A(\cdot)] = D(\lambda) + f(\cdot)N(\lambda) \tag{26}$$

(e) In the case of a linear time-invariant system, the symbolic polynomial is equal to the characteristic polynomial.

4.2 Calculation of the Change of Basis

The calculation of the change of basis can be made *a posteriori*, once the new model $\bar{A}(\cdot)$ has been defined. Here we deal only with $c(1)$ systems. Suppose it has been proved that two matrices $A(\cdot)$ and $\bar{A}(\cdot)$ have the same symbolic polynomial, developed as in Definition 1. Then $\bar{A}$ is developed into

$$\bar{A} = \bar{A}_0 + \bar{b}(\cdot)\bar{c}^T \tag{27}$$

The change-of-basis P is obvious under the condition that all nonconstant terms of $A(\cdot)$ are located in only its last column, and similarly for $\bar{A}(\cdot)$. Then

$$c^T = \bar{c}^T = [0 —— 0\,1] \tag{28}$$

and P is defined by the expression of the development of $\bar{b}(\cdot)$ in terms of $b_i(\cdot)$, as illustrated by the following example:

$$A(\cdot) = \begin{bmatrix} 0 & 0 & -1+\epsilon^2 \\ 1 & 0 & -2+2\epsilon^4 \\ 0 & 1 & -1-\epsilon^2+\epsilon^4 \end{bmatrix},$$

$$B_0(\cdot) = \begin{bmatrix} \epsilon^2 + \sin t \\ 2\epsilon^4 \\ -\epsilon^2+\epsilon^4 \end{bmatrix} \tag{29}$$

Then $b(\cdot)$ can be defined by

$$b_1(\cdot) = \epsilon^2, \quad b_2(\cdot) = 2\epsilon^4, \quad b_3(\cdot) = -\epsilon^2+\epsilon^4 \tag{30}$$

The symbolic polynomial of $A(\cdot)$ is

$$p[\lambda, A(\cdot)] = [\lambda^3 + \lambda^2 + 2\lambda + 1] - b_1(\cdot)[1] - b_2(\cdot)[\lambda] - b_3[\lambda^2] \quad (31)$$

This polynomial is nonfactorizable.

Suppose the expected form of $A(\cdot)$ is

$$\bar{A}(\cdot) = \begin{bmatrix} -1 & 0 & \alpha(\cdot) \\ 0 & -2 & \beta(\cdot) \\ 1 & 1 & \gamma(\cdot) \end{bmatrix} \quad (32)$$

The symbolic polynomials of $A(\cdot)$ and $\bar{A}(\cdot)$ are identical (and the two matrices are therefore similar) iff

$$\begin{aligned} \alpha(\cdot) &= 1 + b_1(\cdot) - b_2(\cdot) + b_3(\cdot) = 1 - \epsilon^4 \\ \beta(\cdot) &= -7 - b_1(\cdot) + 2b_2(\cdot) - 4b_3(\cdot) = -7 + 3\epsilon^2 \\ \gamma(\cdot) &= 2 + b_3(\cdot) = 2 - \epsilon^3 + \epsilon^4 \end{aligned} \quad (33)$$

The corresponding change of basis is defined by the matrix of $b_i(\cdot)$ coefficients in Eqn. (31); hence

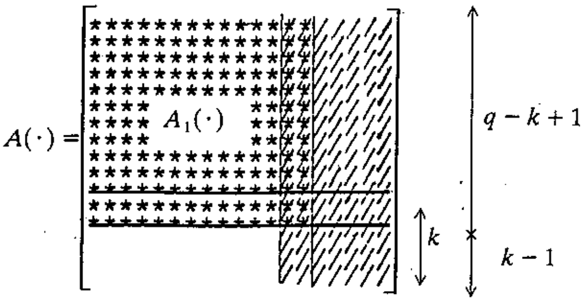
$$P = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -4 \\ 0 & 0 & 1 \end{bmatrix}, \qquad PA(\cdot)P^{-1} = \bar{A}(\cdot) \quad (34)$$

From a knowledge of P, the nonautonomous equations can also be deduced, since $\bar{B}_0(\cdot) = PB_0(\cdot)$.

4.3 Multivariable c(k) and l(k) Types

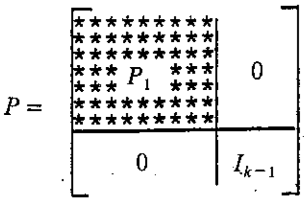
When several columns (or rows) are regrouping the nonconstant elements, two solutions for the generalization of the $c(1)$ or $l(1)$ cases are possible: (a) the definition of a matrix symbolic polynomial, which requires a certain redundancy to be introduced in the order of the representation (Richard 1984) and (b) block partitioning, which allows the above definitions and method to be used. This second solution is developed here.

Let the initial matrix $A(\cdot)$ be of $c(k)$ type, decomposed as



(35)

The nonconstant terms of $A(\cdot)$ are in its k last columns, and a square $(q - k + 1)$ block $A_1(\cdot)$ is isolated, which is of $c(1)$ type. The preceding method can be used for conditioning $A_1(\cdot)$ into $\bar{A}_1(\cdot)$, through a $(q - k + 1)$-ordered change-of-basis P_1. Then this change is included in a qth-order block diagonal change-of-basis P, changing $A(\cdot)$ into $\bar{A}(\cdot)$:



(36)

This procedure allows the *a priori* choice of a large part of the final conditioning $\bar{A}(\cdot)$, but not of the whole matrix $\bar{A}(\cdot)$. However, it can be conducted, as in the $c(1)$ or $l(1)$ case, by computer.

5. Arrow-Form Model

5.1 Definition

The arrow-form matrices appear as a nonlinear generalization of the Jordan form, Eqn. (6), and of modal decomposition of linear systems. In its simplest shape, Eqn. (37), all its nonzero elements are located in its last row, last column and principal diagonal. In the general case, Eqn. (38) (called thick arrow form), this structure is conserved, but rows or diagonal are replaced by the block rows and/or block diagonal:

$$A_A(\cdot) = \begin{bmatrix} -\lambda_1 & & 0 & \beta_1 \\ & \ddots & & \vdots \\ 0 & & -\lambda_{q-1} & \beta_{q-1} \\ \alpha_1 & \cdots & \alpha_{q-1} & \gamma \end{bmatrix} \quad (37)$$

$$A_{TA}(\cdot) = \begin{bmatrix} D_1 & & 0 & B_1 \\ & \ddots & & \vdots \\ 0 & & D_{n-1} & B_{n-1} \\ -A_1 & \cdots & A_{n-1} & G \end{bmatrix} \quad (38)$$

In the case of $c(k)$ (or $l(k)$) matrices, the blocks A_i, D_i (or B_i, D_i) are assumed to be constant. G is of $k \times k$ dimension; D_i are square blocks of any dimension.

The basic property of such a matrix is that its diagonal elements D_i (or $-\lambda_i$ in Eqn. (37)) can be arbitrarily chosen, which constitutes an outstanding advantage over the classical linear modal form. This is due to the presence of the A_i and B_i blocks. In the case $c(1)$ or $l(1)$ forms of Eqn. (37), the expressions of α_i and β_i are directly known as a function of the λ_i and of the symbolic polynomial, provided that this symbolic polynomial is nonfactorizable (which implies that the λ_i are all different):

$$
\begin{aligned}
R(\lambda) &= \prod_{i=1}^{q-1} (\lambda + \lambda_i), \quad \lambda_i \neq \lambda_j, \quad \forall i \neq j \\
p[\lambda, A_{\mathrm{A}}(\cdot)] &= \lambda^q + p_{q-1}(\cdot)\lambda^{q-1} + \cdots + p_0(\cdot) \\
\gamma &= -p_{q-1}(\cdot) + \sum_{i=1}^{q-1} \lambda_i \\
\alpha_i\beta_i &= -\left[\frac{(\lambda+\lambda_i)p[\lambda, A_{\mathrm{A}}(\cdot)]}{R(\lambda)}\right]_{\lambda=-\lambda_i} \neq 0
\end{aligned}
$$

α_i constant in the $c(1)$ case, β_i constant in the $l(1)$ case (39)

It appears that the α_i and β_i can be balanced at will, since only their product is fixed. Moreover, these formulas have been generalized to the case where Eqn. (37) presents a Jordan-type diagonal, which allows the choice of λ_i equal.

The arrow form is very suitable for automatic control problems, since it provides an answer to the four goals presented in Sect. 1: it naturally leads to subsystem partitioning and is a rather hollow matrix, with a possible choice of the diagonal elements. Section 5.2 concerns the possible reduction of the nonconstant elements by means of appropriate λ_i choices. Indeed, choosing the zeros of $p_j(\lambda)$ in Eqn. (23) as $-\lambda_i$ makes the $b_j(\cdot)$ coefficient in $\alpha_i\beta_i$ cancel.

5.2 *Lur'e–Postnikov-Type Systems*

The arrow form is particularly suited to the study of systems of the type presented in Fig. 1. In this case, the symbolic polynomial is given by Eqn. (26), and three particular choices of λ_i are of great interest.

(a) $N(-\lambda_i) = 0$. If $N(s)$ presents $q-1$ real negative distinct zeros $(-\lambda_i)$, all the α_i and β_i are constant in Eqn. (37); the only remaining nonconstant term in Eqn. (37) is then the "tip coefficient" γ.

(b) $D(-\lambda_i) = 0$. If $D(s)$ presents $q-1$ real negative distinct zeros $(-\lambda_i)$, all the α_i $[l(1)]$ or β_i $[c(1)]$ are proportional to $f(\cdot)$.

(c) $D(-\lambda_i) + \rho N(-\lambda_i) = 0, \quad \rho \in \mathbb{R}$. If there exists a real number ρ such that $D(s) + \rho N(s)$ presents $q-1$ real negative distinct zeros $(-\lambda_i)$, then all the α_i $[l(1)]$ or $\beta_i\,[c(1)]$ are proportional to the same nonconstant term $[f(\cdot) - \rho]$.

These choices can obviously be combined at will and extended to the multiple $N(s)$ or $D(s)$ zero case by means of the abovementioned Jordan-type arrow matrix. These three cases allow, for example, a well-adapted stability study by means of quadratic-plus-integral Lyapunov functions together with vector norm functions (Tzafestas 1984).

5.3 *Example*

Computer exploitation of such properties has been made in LIMA (Logiciel Intéractif de Modélisation et d'Analyse) software. LIMA allows

(a) calculation of the symbolic polynomial of any $c(1)$ matrix $A(\cdot)$,

(b) determination of any $c(1)$ arrow form matrix, similar to $A(\cdot)$,

(c) advice on choosing the diagonal parameters,

(d) calculation of the corresponding changes of basis and

(e) stability analysis, using several criteria on the models obtained.

As an illustration, the $c(1)$ matrix $A(\cdot)$ defined in Eqn. (40) can be brought into the arrow form of Eqn. (41):

$$
A(\cdot) = \begin{bmatrix} -5 & 0 & 0 & 0 & b_1(\cdot) \\ 0 & -4 & 0 & 0 & b_2(\cdot) \\ 0.5 & 0.5 & -2 & 0 & 0 \\ 0 & 1 & 0 & -1.5 & 0 \\ 0.5 & 0.5 & 1 & 1 & -2.5 \end{bmatrix} \qquad (40)
$$

$$
\begin{aligned}
p[\lambda, A(\cdot)] &= D(\lambda) + b_1(\cdot)N_1(\lambda) + b_2(\cdot)N_2(\lambda) \\
D(\lambda) &= (\lambda+1.5)(\lambda+2)(\lambda+2.5)(\lambda+4)(\lambda+5) \\
N_1(\lambda) &= -0.5(\lambda+1.5)(\lambda+3)(\lambda+4) \\
N_2(\lambda) &= -0.5(\lambda+1.814)(\lambda+4.686)(\lambda+5)
\end{aligned}
$$

$$
\begin{bmatrix} -1.5 & 0 & 0 & 0 & 0.133b_2(\cdot) \\ 0 & -3 & 0 & 0 & -0.5 + 0.667b_2(\cdot) \\ 0 & 0 & -4 & 0 & -0.3b_2(\cdot) \\ 0 & 0 & 0 & -5 & 0.5b_1(\cdot) \\ 1 & 1 & 1 & 1 & -1.5 \end{bmatrix} \qquad (41)
$$

The change-of-basis P from Eqn. (40) to Eqn. (41) is

$$
P = \begin{bmatrix} 0 & 0.133 & 0 & 0.333 & 0 \\ 0 & 0.667 & 1 & 0.667 & -1 \\ 0 & -3 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (42)
$$

Another arrow form of Eqn. (40) could be

$$
\begin{bmatrix} -1.5 & 0 & 0 & 0 & 0.7b_2(\cdot) \\ 0 & -1.814 & 0 & 0 & -0.14 + 0.2b_1(\cdot) \\ 0 & 0 & -4 & 0 & -0.2b_2(\cdot) \\ 0 & 0 & 0 & -4.686 & 0.64 + 0.29b_1(\cdot) \\ 1 & 1 & 1 & 1 & -3 \end{bmatrix} \qquad (43)
$$

See also: State-Space Modelling: Square Root Algorithms; State-Space Self-Tuners; State-Space Techniques: History

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