

Design of interval observers for uncertain distributed systems

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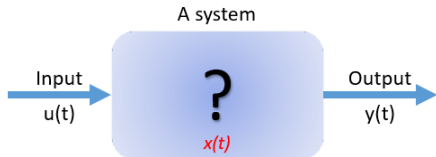
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Introduction: *Estimation and control*

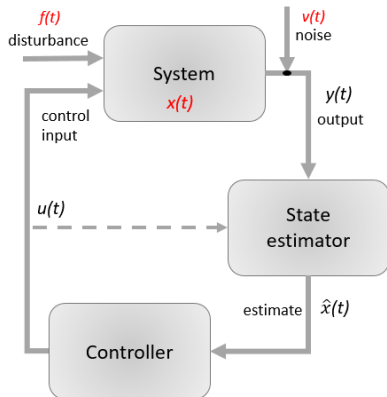
What is estimation?



Uncertainties:

- Parameters,
- Measurement noise,
- External disturbance,
- Initial conditions...

Analysis & Design

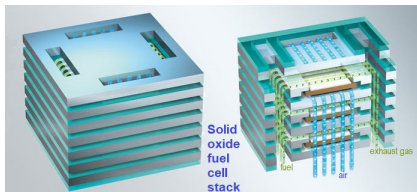


- System model
- Performance & uncertainty
- Adaptive, robust, *interval*...

Motivation: *Partial differential equations (PDEs)*



Tubular plug flow reactor



Dynamic systems:

- Ordinary differential equations (ODEs)
- Delay differential equations (DDEs)
- **PDEs:**
 - Infinite-dimensional spaces
 - Boundary conditions (**Dirichlet**, Neumann, mixed)
 - **Parabolic**, hyperbolic, etc
 - Different topology

Distribution in space

- * sound
- * heat
- * fluid flow
- * electrostatics
- * quantum mechanics, etc

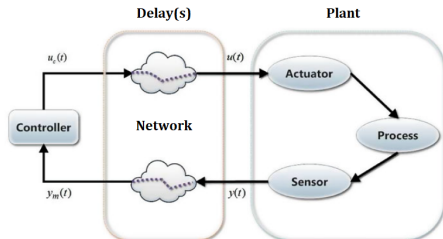
Motivation: *Delay differential equations (DDEs)*

Time-delays:

- * transportation
- * communication networks
- * chemical control systems
- * laser models
- * biology, etc

Challenges:

- Delay may be a source of *instability* and oscillations
- Time-delay systems are *infinite-dimensional*
- Delays in different stages: *in state, measurements, control input*
- Positivity of solutions

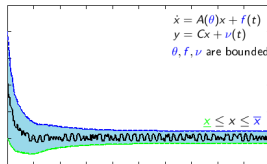
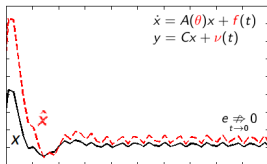
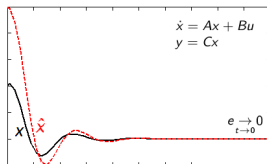


Introduction: *Interval estimation*

- ✓ Conventional observer \implies estimation error $e \rightarrow 0$
- ✗ **Uncertainties** \implies in many cases estimation error $e \neq 0$
- ★ **Interval observer** \implies a set of admissible values (*an interval*) for the state

The main requirement: *Positivity*

- Metzler matrix $A = \begin{bmatrix} \cdot & + \\ + & \cdot \end{bmatrix}$ ($\forall_{i \neq j} a_{ij} \geq 0$)
- Estimation errors $\underline{e} = x - \underline{x} \geq 0, \bar{e} = \bar{x} - x \geq 0$



Interval observers: *State of the art*

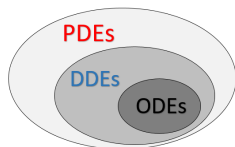
Numerous extensions for various classes of dynamical models:

- for linear systems [Gouzé et al. 2000, Mazenc and Bernard, 2010, 2011],
- nonlinear [Raïssi et al. 2010, 2012],
- discrete-time [Mazenc et al., 2012a, 2013, 2014, Efimov et al., 2013a,b],
- LPV-systems [Chebotarev et al. 2015, Efimov et al. 2011; 2013],
- time-delay [Mazenc et al., 2012b, Efimov et al., 2013c, 2016, 2015a],
- descriptor systems [Efimov et al. 2015], etc.

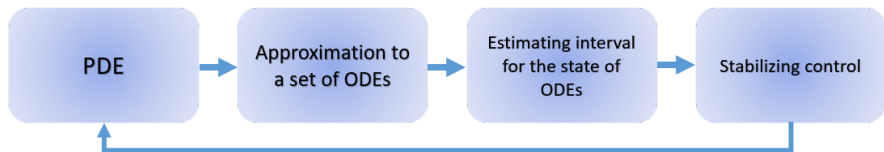
The *primary objective* was to apply the method of interval observers for distributed parameter systems, which are described by PDEs:

1. by using approximation to ODEs [Wheeler 1973],
2. distributed interval observer;
- & develop control based on interval observer [Efimov et al. 2013].

→ 3. In addition, an interval observer design for a *second order DDE* with delayed measurements.



1. Approximation approach



A considered parabolic PDE with an output

$$\begin{aligned}\rho(x) \frac{\partial z(x, t)}{\partial t} &= L[x, z(x, t)] + r(x, t) \\ \forall (x, t) &\in I \times (0, T), \\ z(x, 0) &= z_0(x) \quad \forall x \in I, \\ 0 = z(0, t) &= z(1, t) \quad \forall t \in (0, T),\end{aligned}\quad (1)$$

where $I = [0, 1]$ and $T > 0$,

$$L(x, z) = \frac{\partial}{\partial x} \left(a(x) \frac{\partial z}{\partial x} \right) - b(x) \frac{\partial z}{\partial x} - q(x)z,$$

$a, b, q, \rho \in L^\infty(I, \mathbb{R})$ and $\exists a_0, a_1, \rho_0, \rho_1 \in \mathbb{R}_+$
s.t.

$$\begin{aligned}0 < a_0 \leq a(x) \leq a_1, \quad 0 < \rho_0 \leq \rho(x) \leq \rho_1 \\ \forall x \in I,\end{aligned}$$

and $a', b' \in L^2(I, \mathbb{R})$, where $a' = \partial a(x) / \partial x$.

Assume that the state $z(x, t)$ is available for **measurements in certain points** $x_i^m \in I$ for $1 \leq i \leq p$:

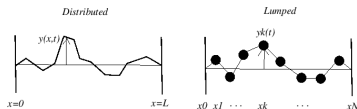
$$y_i(t) = z(x_i^m, t) + \nu_i(t),$$

where $y(t), \nu(t) \in \mathbb{R}^p$, $\nu \in L^\infty(\mathbb{R}_+, \mathbb{R}^p)$ is the measurement noise.

We also assume that in (1)

$$r(x, t) = \sum_{k=1}^p r_{1k}(x) u_k(t) + r_0(x, t),$$

where $u(t) \in \mathbb{R}^p$ is a control (known input), $r_{1k} \in L^\infty(I, \mathbb{R})$ and $r_0 \in L^\infty(I \times [0, T], \mathbb{R})$.



Approximation

The continuous-time Galerkin approximation $Z(\cdot, t) \in M$ to the solution $z(x, t)$ of the parabolic system (1) is defined by

$$\begin{aligned} \left(\rho \frac{\partial Z}{\partial t}, V \right) &= \mathcal{L}(Z, V) + (r, V) \quad \forall V \in M, \quad \forall t \in (0, T); \\ \mathcal{L}(Z - z_0, V) - \lambda(Z - z_0, V) &= 0 \quad \forall V \in M, \quad t = 0. \end{aligned} \quad (2)$$

Then the solution $Z(x, t)$ of (2) can be presented as

$$Z(x, t) = \sum_{j=1}^N \xi_j(t) \Phi_j(x),$$

where $\xi = [\xi_1 \dots \xi_N]^T \in \mathbb{R}^N$ is the **vector of coefficients** satisfying the ordinary differential equations for all $1 \leq j \leq N$, which finally can be presented in the form:

$$\dot{\xi}(t) = A\xi(t) + G\bar{r}(t) \quad \text{a.a. } t \in (0, T), \quad \xi(0) = \xi_0, \quad (3)$$

where $G\bar{r}(t) = Bu(t) + Gd(t)$. As for **output**: for a finite elements approximation we can assign

$$y_i(t) = Z(x_i^m, t) + \nu_i(t) + e_i(t),$$

which can be finally presented like

$$y(t) = C\xi(t) + v(t). \quad (4)$$

Error of approximation

ASSUMPTION 1.1. *Boundedness of z and its derivative*

$\exists s > 0$, $l_1 > 0$ and $l_2 > 0$ s. t. the solution z of (1) belongs to $L^\infty([0, T], W^{s+1, \infty}(I, \mathbb{R}))$ and $\partial z / \partial t \in L^2([0, T], H^{s+1}(I, \mathbb{R}))$,

$$\|z\|_{L^\infty([0, T], W^{s+1, \infty}(I, \mathbb{R}))} \leq l_1,$$

$$\|\partial z / \partial t\|_{L^2([0, T], H^{s+1}(I, \mathbb{R}))} \leq l_2.$$

PROPOSITION 1.1. [1] *Error of approximation*

Let Assumption 1.1 be satisfied, then there is $\varrho > 0$ s. t.

$$\|Z - z\|_{L^\infty(I \times (0, T), \mathbb{R})} \leq \varrho h^{s+1} (l_1 + l_2),$$

where z and Z are the solutions of (1) and (2), respectively.

[1] Wheeler, M. (1973).

Interval estimates

LEMMA 1.1. *Relations between estimates for ξ and Z*

Let $\exists \underline{\xi}, \bar{\xi} \in L^\infty([0, T], \mathbb{R}^N)$ s.t. for solution ξ of (3) we have

$$\underline{\xi}(t) \leq \xi(t) \leq \bar{\xi}(t) \quad \forall t \in [0, T],$$

then for solution Z of (2),

$$\underline{Z}(x, t) \leq Z(x, t) \leq \bar{Z}(x, t) \quad (5) \\ \forall (x, t) \in I \times [0, T]$$

and $\underline{Z}, \bar{Z} \in L^\infty(I \times [0, T], \mathbb{R})$, where

$$\underline{Z}(x, t) = \sum_{j=1}^N (\underline{\xi}_j(t) \Phi_j^+(x) - \bar{\xi}_j(t) \Phi_j^-(x)),$$

$$\bar{Z}(x, t) = \sum_{j=1}^N (\bar{\xi}_j(t) \Phi_j^+(x) - \underline{\xi}_j(t) \Phi_j^-(x)).$$

LEMMA 1.2. *Relations between estimates for Z and z*

Let Assumption 1.1 be satisfied and $\exists \underline{Z}, \bar{Z} \in L^\infty(I \times [0, T], \mathbb{R})$ s.t. (5) be true for solution Z of (2), then there is $\varrho > 0$ s.t. for solution z of (1),

$$\underline{z}(x, t) \leq z(x, t) \leq \bar{z}(x, t) \quad (6) \\ \forall (x, t) \in I \times [0, T]$$

and $\underline{z}, \bar{z} \in L^\infty(I \times [0, T], \mathbb{R})$, where

$$\underline{z}(x, t) = \underline{Z}(x, t) - \varrho h^{s+1} (l_1 + l_2),$$

$$\bar{z}(x, t) = \bar{Z}(x, t) + \varrho h^{s+1} (l_1 + l_2).$$

Assumptions

Assumption 1.2. *Bounds for model uncertainty*

Let $\underline{z}_0 \leq z_0 \leq \bar{z}_0$ for some known $\underline{z}_0, \bar{z}_0 \in L^\infty(I, \mathbb{R})$, let also two functions $\underline{r}_0, \bar{r}_0 \in L^\infty(I \times [0, T], \mathbb{R})$ and a constant $\nu_0 > 0$ be given such that

$$\underline{r}_0(x, t) \leq r_0(x, t) \leq \bar{r}_0(x, t), \quad |\nu(t)| \leq \nu_0 \quad \forall (x, t) \in I \times (0, T).$$

Assumption 1.3. *Similarity transformation*

There are a matrix $L \in \mathbb{R}^{N \times p}$ and a Metzler matrix $D \in \mathbb{R}^{N \times N}$ such that the matrices $A - LC$ and D have the same eigenvalues and the pairs $(A - LC, \chi_1)$ and (D, χ_2) are observable for some $\chi_1 \in \mathbb{R}^{1 \times N}$, $\chi_2 \in \mathbb{R}^{1 \times N}$.

Coordinates transformation: According to Assumption 1.3 and [2] there is a nonsingular transformation matrix $S \in \mathbb{R}^{N \times N}$ such that $D = S(A - LC)S^{-1}$. In the new coordinates $\zeta = S\xi$ the system (3) takes the form:

$$\dot{\zeta}(t) = D\zeta(t) + SBu(t) + SLy(t) + \delta(t), \quad \delta(t) = S[Gd(t) - Lv(t)]. \quad (7)$$

[2] Raïssi, T., Efimov, D., and Zolghadri, A. (2012).

Interval observer

For the system (7) an **interval observer** can be proposed:

$$\begin{aligned}\dot{\underline{\zeta}}(t) &= D\underline{\zeta}(t) + SBu(t) + SLy(t) + \underline{\delta}(t), \\ \dot{\overline{\zeta}}(t) &= D\overline{\zeta}(t) + SBu(t) + SLy(t) + \overline{\delta}(t), \\ \underline{\zeta}(0) &= S^+\underline{\xi}_0 - S^-\overline{\xi}_0, \quad \overline{\zeta}(0) = S^+\overline{\xi}_0 - S^-\underline{\xi}_0, \\ \underline{\xi}(t) &= (S^{-1})^+\underline{\zeta}(t) - (S^{-1})^-\overline{\zeta}(t), \\ \overline{\xi}(t) &= (S^{-1})^+\overline{\zeta}(t) - (S^{-1})^-\underline{\zeta}(t).\end{aligned}\tag{8}$$

PROPOSITION 1.2 Let assumptions 1.1, 1.2 and 1.3 be satisfied. Then for (3), (4) with the interval observer (8) the relations

$$\underline{\xi}(t) \leq \xi(t) \leq \overline{\xi}(t) \quad \forall t \in [0, T].\tag{9}$$

are satisfied. In addition, $\underline{\xi}, \overline{\xi} \in C([0, T], \mathbb{R}^N)$ if $A - LC$ is Hurwitz.

Control based on interval observer

In our work, the control is chosen as a conventional state linear feedback:

$$u(t) = \underline{K}\underline{\zeta}(t) + \overline{K}\overline{\zeta}(t) \quad (10)$$

where $\underline{K}, \overline{K} \in \mathbb{R}^{m \times N}$ are two feedback matrix gains to be designed. To this end, let us consider the combined system, which consists of (7), (8) and (10):

$$\dot{\eta}(t) = (\tilde{A} + \tilde{B}K\Gamma)\eta(t) + \tilde{\Delta}(t), \quad (11)$$

where $\eta = [\zeta^T \ \underline{\zeta}^T \ \overline{\zeta}^T]^T \in \mathbb{R}^{3N}$ is the combined state and

$$\tilde{A} = \begin{bmatrix} SAS^{-1} & 0 & 0 \\ SLCS^{-1} & D & 0 \\ SLCS^{-1} & 0 & D \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} SB \\ SB \\ SB \end{bmatrix}, \quad K = [0 \ \underline{K} \ \overline{K}],$$
$$\Gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_N & 0 \\ 0 & 0 & I_N \end{bmatrix}, \quad \tilde{\Delta}(t) = \begin{bmatrix} SGd(t) \\ \underline{\delta}(t) + SLv(t) \\ \overline{\delta}(t) + SLv(t) \end{bmatrix}.$$

Control based on interval observer

PROPOSITION 1.3 Let assumptions 2 and 3 be satisfied. Then for (3), (4) with the interval observer (8) and control law (10) the relations (9) are satisfied. In addition, $\xi, \underline{\xi}, \bar{\xi} \in L^\infty([0, T], \mathbb{R}^N)$ if there exists a matrix $\tilde{X} \in \mathbb{R}^{3N \times 3N}$ in the form

$$\tilde{X} = \begin{bmatrix} X_P & 0 & 0 \\ 0 & & \\ 0 & X_Q & \end{bmatrix}, \quad X_P = X_P^T \in \mathbb{R}^{N \times N}, \quad X_Q = X_Q^T \in \mathbb{R}^{2N \times 2N}$$

and $\tilde{Y} \in \mathbb{R}^{m \times 2N}$ that satisfy the following matrix inequalities

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T + \tilde{B} \begin{bmatrix} 0 & \tilde{Y} \end{bmatrix} + \begin{bmatrix} 0 & \tilde{Y} \end{bmatrix}^T \tilde{B}^T \prec 0, \quad X_P \succ 0, \quad X_Q \succ 0, \quad (12)$$

then $[\underline{K} \ \bar{K}] = \tilde{Y}X_Q^{-1}$ and the system (3), (4), (8) and (10) is stable.

Example 1: Black Scholes model

$$\frac{\partial \mathcal{V}(S, t)}{\partial t} = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \mathcal{V}(S, t)}{\partial S^2} - (r(t) - q(t))S \frac{\partial \mathcal{V}(S, t)}{\partial S} + r(t)\mathcal{V}(S, t) + g(S) \quad \forall (S, t) \in I \times (0, T),$$

$$\mathcal{V}(S, 0) = \mathcal{V}_0 \quad \forall S \in I,$$

$$0 = \mathcal{V}(0, t) = \mathcal{V}(1, t) \quad \forall t \in (0, T).$$

- $\mathcal{V}(S, t)$ is the price of the option

- S is the stock price belonging to a given interval of admissible prices I
- r is the risk-free interest rate
- q is the dividend rate of the underlying asset
- σ is the volatility of the stock
- $g(S)$ is an inhomogeneous term

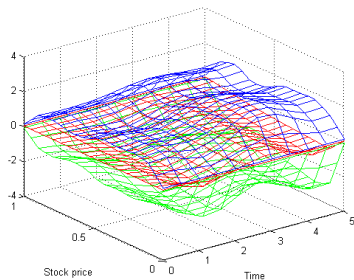


Figure 1: The results of interval estimation for Black-Scholes model

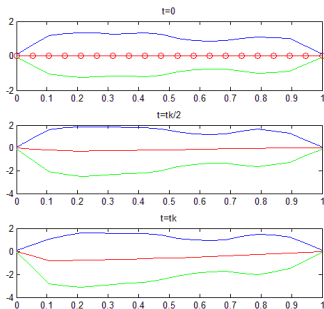


Figure 2: The results of interval estimation for the Black Scholes model at time $t = 0$, $t = \frac{T}{2}$, $t = T$

Example 2: Unstable system

Consider an **unstable** academic example of (1) for

$$\begin{aligned}\rho(x) &= 0.7 \sin(0.67x), \quad a(x) = 1.5 + 1.5\cos(0.2x^{0.25}), \\ b(x) &= -2 + \sin(2\sqrt{x}), \quad q(x) = -0.8 - x^2 \cos(3x), \quad r_1(x) = x^3 + 2.5, \\ r_0(x, t) &= r_{01}(x)r_{02}(t), \quad r_{01}(x) = 0.1 \cos(3\pi x), \quad |r_{02}(t)| \leq 1,\end{aligned}$$

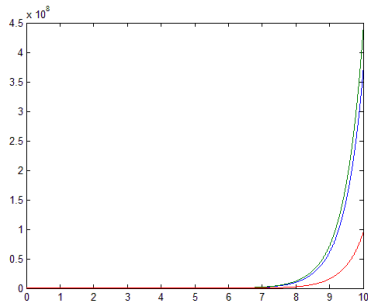


Figure 3: The output of **unstable** academic example for $N = 10$

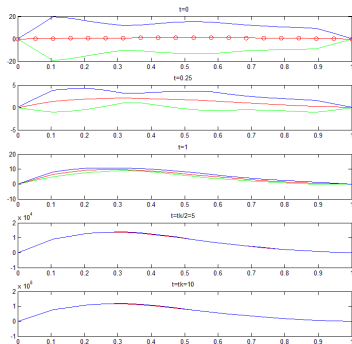


Figure 4: The results of interval estimation of academic example for different instants of time: $t = 0, 0.25, 1, 5, 10$ for $N = 10$

Example 2: Stabilization

To calculate the control matrix $[K \ \bar{K}]$ the LMIs (12) were used with YALMIP optimization toolbox in Matlab, and it was found

$$\underline{K} = \bar{K} = [-0.034 \ 0.126 \ 0.122 \ 0.076 \ 0.185 \ -0.018 \ -0.508 \ -0.022].$$

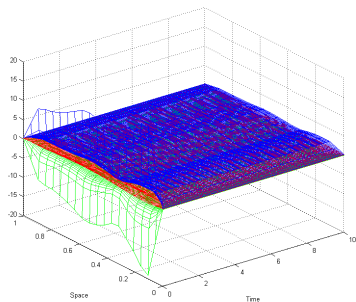


Figure 5: The results of interval estimation of **unstable** academic example for $N = 10$

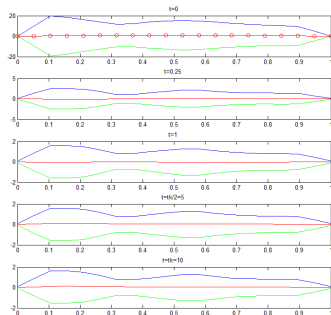
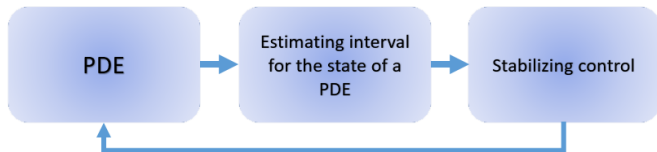


Figure 6: The results of interval estimation of academic example for different instants of time: $t = 0, 0.25, 1, 5, 10$ for $N = 10$

2. Distributed Interval Observer



A considered heat equation and an output

$$\begin{aligned} \frac{\partial z(x, t)}{\partial t} &= L[x, z(x, t)] + r(x, t) + \\ &+ \sum_{j=0}^p b_j(x) u_j(t) \quad \forall (x, t) \in I \times \mathcal{T}, \\ z(x, t_0) &= z_0(x) \quad \forall x \in I, \\ z(0, t) &= \alpha(t), \quad z(\ell, t) = \beta(t) \quad \forall t \in \mathcal{T}, \end{aligned} \quad (13)$$

where $I = [0, \ell]$ with $0 < \ell < +\infty$, $\mathcal{T} = [t_0, t_0 + T)$ for $t_0 \in \mathbb{R}$ and $T > 0$,

$$L(x, z) = \frac{\partial}{\partial x} \left(a(x) \frac{\partial z}{\partial x} \right) + q(x)z,$$

$a \in H^2(I, \mathbb{R})$, $q \in H^1(I, \mathbb{R})$ and there exist $a_{\min}, a_{\max} \in \mathbb{R}_+$ such that

$$0 < a_{\min} \leq a(x) \leq a_{\max} \quad \forall x \in I.$$

Assume that the state $z(x, t)$ is available for measurements in certain points $0 \leq x_1^m < x_2^m < \dots < x_p^m \leq \ell$:

$$y_i(t) = z(x_i^m, t) + \nu_i(t), \quad i = \overline{1, p},$$

where $y(t) = [y_1(t), \dots, y_p(t)]^T \in \mathbb{R}^p$ is the measured output signal, $\nu(t) = [\nu_1(t), \dots, \nu_p(t)] \in \mathbb{R}^p$ and $\nu \in L^\infty(\mathbb{R}_+, \mathbb{R}^p)$ is the measurement noise.

Stability analysis: ISS

PROPOSITION 2.1 Let $\alpha, \beta \in H^1(\mathcal{T}, \mathbb{R})$ and $a_{\min} \frac{\pi^2}{\ell^2} = q_{\max} + \chi$, where $\chi > 0$ and $q_{\max} = \sup_{x \in I} q(x)$, then for solutions of (13) the following estimate is satisfied for all $t \in \mathcal{T}$:

$$\begin{aligned} \frac{1}{2} \int_0^\ell z^2(x, t) dx &\leq e^{-\chi(t-t_0)} \int_0^\ell w_0^2(x) dx + \chi^{-2} \int_0^\ell \tilde{r}^2(x, t) dx \\ &\quad + \frac{\ell}{2} [\alpha^2(t) + \beta^2(t)], \end{aligned}$$

where $w_0(x) = z_0(x) - \delta(x, t_0)$, $\delta(x, t) = \alpha(t) + \frac{x}{\ell}(\beta(t) - \alpha(t))$ and

$$\begin{aligned} \tilde{r}(x, t) &= r(x, t) + \frac{1}{\ell} \frac{\partial a(x)}{\partial x} (\beta(t) - \alpha(t)) \\ &\quad + q(x) \delta(x, t) - \delta_t(x, t). \end{aligned}$$

The main restriction of that proposition is:

$$a_{\min} \frac{4\pi^2}{\ell^2} > q_{\max}.$$

Condition for positivity of solutions [3]

DEFINITION The distributed parameter system (13) is called nonnegative (positive) on the interval \mathcal{T} if for

$$\alpha(t) \geq 0, \beta(t) \geq 0, r(x, t) \geq 0 \quad \forall (x, t) \in I \times \mathcal{T}$$

the implication $z_0(x) \geq 0 \Rightarrow z(x, t) \geq 0$ ($z_0(x) > 0 \Rightarrow z(x, t) > 0$) holds for all $(x, t) \in I \times \mathcal{T}$ and for all $z_0 \in H^{-1}(I, \mathbb{R})$.

PROPOSITION 2.2 Let $\alpha, \beta \in L^2(\mathcal{T}, \mathbb{R}_+)$, $r \in L^2(I \times \mathcal{T}, \mathbb{R}_+)$ and $z_0 \in H^{-1}(I, \mathbb{R}_+)$, then

$$z(x, t) \geq 0 \quad \forall (x, t) \in I \times \mathcal{T},$$

i.e. (13) is nonnegative on the interval \mathcal{T} .

[3] Nguyễn and Coron (2016).

Assumptions

ASSUMPTION 2.1. *Uncertainties*

Let $\underline{z}_0 \leq z_0 \leq \bar{z}_0$ for some known $\underline{z}_0, \bar{z}_0 \in H^{-1}(I, \mathbb{R})$, let also functions $\underline{\alpha}, \bar{\alpha}, \underline{\beta}, \bar{\beta} \in L^2(\mathcal{T}, \mathbb{R})$, $\underline{r}, \bar{r} \in L^2(I \times \mathcal{T}, \mathbb{R})$ and a constant $\nu_0 > 0$ be given such that for all $(x, t) \in I \times \mathcal{T}$:

$$\begin{aligned}\underline{\alpha}(t) &\leq \alpha(t) \leq \bar{\alpha}(t), \quad \underline{\beta}(t) \leq \beta(t) \leq \bar{\beta}(t), \\ \underline{r}(x, t) &\leq r(x, t) \leq \bar{r}(x, t), \quad |\nu(t)| \leq \nu_0.\end{aligned}$$

ASSUMPTION 2.2. *Differentiability of noise and boundary conditions*

Let $\alpha, \beta \in H^1(\mathcal{T}, \mathbb{R})$ and $\nu \in H^1(\mathcal{T}, \mathbb{R}^p)$.

The form of the interval observer

The interval observer for (13) is as follows for $i = 0, 1, \dots, p$:

$$\begin{aligned}\frac{\partial \bar{z}(x, t)}{\partial t} &= L[x, \bar{z}(x, t)] + \bar{r}(x, t) \quad \forall (x, t) \in I_i \times \mathcal{T}, \\ \bar{z}(x, t_0) &= \bar{z}_0(x) \quad \forall x \in I_i, \\ \bar{z}(x_i^m, t) &= \bar{Z}_i(t), \quad \bar{z}(x_{i+1}^m, t) = \bar{Z}_{i+1}(t) \quad \forall t \in \mathcal{T}; \\ \frac{\partial \underline{z}(x, t)}{\partial t} &= L[x, \underline{z}(x, t)] + \underline{r}(x, t) \quad \forall (x, t) \in I_i \times \mathcal{T}, \\ \underline{z}(x, t_0) &= \underline{z}_0(x) \quad \forall x \in I_i, \\ \underline{z}(x_i^m, t) &= \underline{Z}_i(t), \quad \underline{z}(x_{i+1}^m, t) = \underline{Z}_{i+1}(t) \quad \forall t \in \mathcal{T},\end{aligned}\tag{14}$$

where $\bar{z} \in C^0(\mathcal{T}, H^{-1}(I, \mathbb{R}))$ and $\underline{z} \in C^0(\mathcal{T}, H^{-1}(I, \mathbb{R}))$ are upper and lower estimates of the solution $z(x, t)$. The upper and lower estimates for the **boundary conditions** are

$$\begin{aligned}\bar{Z}(t) &= [\bar{\alpha}(t), y_1(t) + \nu_0, \dots, y_p(t) + \nu_0, \bar{\beta}(t)]^T, \\ \underline{Z}(t) &= [\underline{\alpha}(t), y_1(t) - \nu_0, \dots, y_p(t) - \nu_0, \underline{\beta}(t)]^T.\end{aligned}$$

The domain I of the solution of (13) is divided on $p + 1$ subdomains $I_i = [x_i^m, x_{i+1}^m]$ with $x_0^m = 0$ and $x_{p+1}^m = \ell$.

THEOREM 2.1 Let assumptions 2.1 and 2.2 be satisfied, then in (13), (14):

$$\underline{z}(x, t) \leq z(x, t) \leq \bar{z}(x, t) \quad \forall (x, t) \in I \times \mathcal{T}.$$

In addition, if

$$\Delta x^m < 2\pi \sqrt{\frac{a_{\min}}{q_{\max}}}, \quad (15)$$

where $\Delta x^m = \max_{0 \leq i \leq p} x_{i+1}^m - x_i^m$, then for all $t \in \mathcal{T}$:

$$\begin{aligned} \|\bar{z}(\cdot, t) - z(\cdot, t)\|_{L^2(I, \mathbb{R})}^2 &\leq 4e^{-\chi(t-t_0)} [\|\bar{z}_0 - z_0\|_{L^2(I, \mathbb{R})}^2 + \bar{\varrho}(t_0)] \\ &\quad + 8\chi^{-2} \|\bar{r}(\cdot, t) - r(\cdot, t)\|_{L^2(I, \mathbb{R})}^2 + \bar{\gamma}(t), \\ \|z(\cdot, t) - \underline{z}(\cdot, t)\|_{L^2(I, \mathbb{R})}^2 &\leq 4e^{-\chi(t-t_0)} [\|z_0 - \underline{z}_0\|_{L^2(I, \mathbb{R})}^2 + \underline{\varrho}(t_0)] \\ &\quad + 8\chi^{-2} \|r(\cdot, t) - \underline{r}(\cdot, t)\|_{L^2(I, \mathbb{R})}^2 + \underline{\gamma}(t). \end{aligned}$$

Stabilizing control based on distributed observer

The main restriction on stability for the system (13) is $q_{\max} < a_{\min} \frac{\pi^2}{\ell^2}$, as well as the inequality (15) for the IO. To overcome this, let's consider

$$u_j(t) = -\frac{K_j}{\Delta x^m} \int_{x_j^m}^{x_{j+1}^m} (z(\xi, t) + \bar{z}(\xi, t)) d\xi, \quad j = 0, \dots, p, \quad (16)$$

Highlights for analysis:

- ★ The same number of sensors and actuators with collocated subintervals I_j , the shape function

$$\begin{cases} b_j(x) = 0 & x \notin I_j, \\ b_j(x) = 1 & x \in I_j. \end{cases}$$

- ★ Shifting boundary conditions $\delta(x, t) = \alpha(t) + \frac{x}{\ell}(\beta(t) - \alpha(t))$, $w(x, t) = z(x, t) - \delta(x, t)$.
- ★ Lyapunov function: $V(t) = \int_0^\ell w^2(x, t) dx$.
- ★ Poincaré's and Young's inequalities.

THEOREM 2.2 Let assumptions 2.1 and 2.2 be satisfied. Let there exist $\kappa > 0$, $K > 0$, $\gamma > 0$ and $\Delta x^m < \pi \sqrt{\frac{a_{\min}}{q_{\max}}}$ that satisfy the LMI

$$\Phi \triangleq \begin{bmatrix} 2(\kappa + q_{\max} - 2K) & 2K & 1 \\ 2K & \frac{-2a_{\min}\pi^2}{(\Delta x^m)^2} & 0 \\ 1 & 0 & -\gamma^2 \end{bmatrix} \preceq 0.$$

Then for the solutions of system (13) with (16), the interval inclusion

$$\underline{z}(x, t) \leq z(x, t) \leq \bar{z}(x, t) \quad \forall (x, t) \in I \times \mathcal{T}.$$

and the estimates on $\|z(\cdot, t) - \underline{z}(\cdot, t)\|_{L^2(I, \mathbb{R})}$, $\|\bar{z}(\cdot, t) - z(\cdot, t)\|_{L^2(I, \mathbb{R})}$ from Theorem 2.1 are valid and

$$\begin{aligned} \frac{1}{2} \int_0^\ell z^2(x, t) dx &\leq e^{-2\kappa(t-t_0)} \int_0^\ell w_0^2(x) dx + \frac{\gamma^2}{2\kappa} \int_0^\ell R(x, t)^2 dx \\ &+ \frac{\ell}{2} (\alpha^2(t) + \beta^2(t)) \quad \forall (x, t) \in I \times \mathcal{T}. \end{aligned}$$

Example 3: Unstable heat-equation (13)

$$a(x) = \frac{1}{4}(1 + \frac{3}{4} \sin(2\pi x)), \quad q(x) = 5 + \frac{1}{2} \cos(\pi x), \quad r(x, t) = \sin(\pi x)[\cos(2t) + \epsilon(t)],$$

$$|\epsilon(t)| \leq 1, \quad \underline{r}(x, t) = \sin(\pi x)[\cos(2t) - 1], \quad \bar{r}(x, t) = \sin(\pi x)[\cos(2t) + 1],$$

$$\underline{z}_0(x) = z_0(x) - 1, \quad \bar{z}_0(x) = z_0(x) + 1,$$

$$\underline{\alpha}(t) = \sin(2t) - 1, \quad \bar{\alpha}(t) = \sin(2t) + 1, \quad \underline{\beta}(t) = \sin(5t) - 1, \quad \bar{\beta}(t) = \sin(5t) + 1.$$

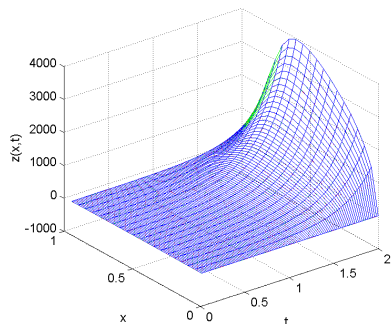


Figure 7: The output of unstable academic example for $N = 20$

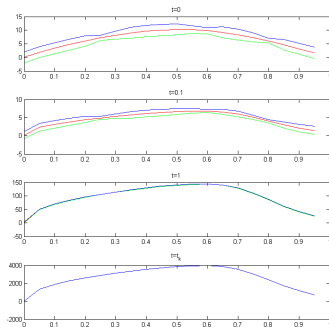


Figure 8: The results of interval estimation for different instants of time: $t = 0, 0.5, 1, 2$ for $N = 20$

Example 3: Stabilization

With these parameters, q_{\max} is larger than $a_{\min} \frac{\pi^2}{\ell^2}$, which means, according to the Proposition 4, that (13) is unstable.

$\Delta x^m = 0.3 \implies$ the restriction for the IO (15) is satisfied.

The control gain $K = 3.2865$ was calculated by Theorem 2.2.

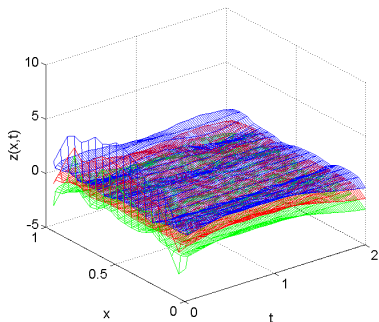


Figure 9: The results of interval estimation of **unstable** academic example for $N = 20$

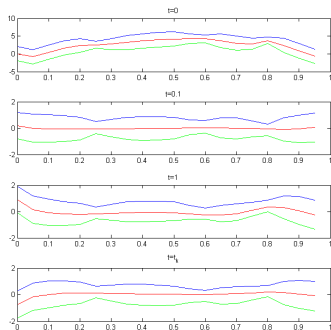


Figure 10: The results of interval estimation for different instants of time: $t = 0, 0.5, 1, 2$ for $N = 20$

Comparison between two approaches

A non-homogeneous heat equation (13) with zero boundary conditions was considered for both methods.

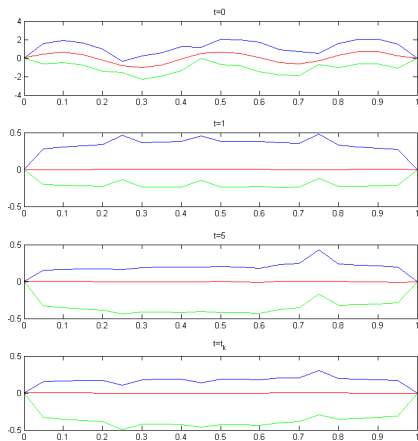


Figure 11: Distributed interval observer approach

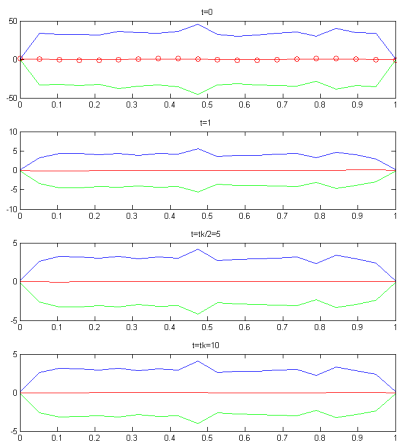


Figure 12: Approximation approach

3. Interval observer for delayed system



Problem statement

A **motivation example** for this part is an unstable second order delayed system with delayed measurements (a delayed model for motion of a single mass point):

$$\ddot{x}(t) = -a(t)x(t - \tau) + f(t), \quad t \geq 0, \quad (17)$$

$$y(t) = x(t - \theta) + \nu(t), \quad \tau \leq \theta. \quad (18)$$

We assume that the parameter $a(t)$ is known and the instant values of $f(t)$ and $\nu(t)$ are unavailable.

Assumption 3.1: Initial conditions

There exist known functions $\underline{\phi}, \bar{\phi} \in \mathcal{C}_\theta$ such that $\underline{\phi}(h) \leq \phi(h) \leq \bar{\phi}(h)$ for all $h \in [-\theta, 0]$, and $\underline{\dot{x}}_0 \leq \dot{x}(0) \leq \bar{\dot{x}}_0$ for some known $\underline{\dot{x}}_0, \bar{\dot{x}}_0 \in \mathbb{R}$.

Assumption 3.2: Noise and input

There exist known signals $\underline{f}, \bar{f} \in \mathcal{L}_\infty$ and a constant $\nu_0 > 0$ such that $\underline{f}(t) \leq f(t) \leq \bar{f}(t)$ and $|\nu(t)| \leq \nu_0$ for all $t \geq 0$.

In a state-space form:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^{=A_0} x(t) + \begin{bmatrix} 0 & 0 \\ -a & 0 \end{bmatrix}^{=A_1} x(t - \tau) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t - \theta) + \nu(t)$$

A_1 is affected by output injection when $\tau = \theta$

Tools: *Delay-independent positivity* [4]

Consider a time-invariant linear system with time-varying delay:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau(t)) + Bf(t), \quad t \in [0, +\infty), \quad (19)$$

$$x(h) = \phi(h) \text{ for } -\bar{\tau} \leq h \leq 0, \quad \phi \in \mathcal{C}_{\bar{\tau}}^n, \quad (20)$$

- $x(t) \in \mathbb{R}^n$, $x_t \in \mathcal{C}_{\bar{\tau}}^n$;
- $\tau : \mathbb{R}_+ \rightarrow [-\bar{\tau}, 0]$ is the time-varying delay, a Lebesgue measurable function of time;
- $\bar{\tau} \in \mathbb{R}_+$ is the maximum delay;
- $f \in \mathcal{L}_{\infty}^m$;
- $\phi : (-\bar{\tau}, 0] \rightarrow \mathbb{R}^n$ is a bounded function of initial conditions.

The system (19) is called **positive** if for $\phi \in \mathcal{C}_{\bar{\tau}}^n$ it has the corresponding solution $x(t) \geq 0$ for all $t \geq 0$.

Lemma 3.1

The system (19) is positive iff A_0 is Metzler, $A_1 \geq 0$ and $Bf(t) \geq 0$ for all $t \geq 0$. A positive system (19) is asymptotically stable with $f(t) \equiv 0$ for all $t \in \mathbb{R}_+$ iff there are $p, q \in \mathbb{R}_+^n$ ($p > 0$ and $q > 0$) such that

$$p^T [A_0 + A_1] + q^T = 0.$$

[4] Haddad et al. 2004, Ait Rami 2009.

Tools: Representation of the solution

Let us define $\mathcal{T} = \{s \in (0, \bar{\tau}] : s \leq \tau(s)\}$, then the system (19), (20) can be rewritten with the same solution $x(t)$ for all $t \geq 0$:

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t - \tau(t)) + Bf(t) + f^*(t), \quad t \in [0, +\infty), \\ x(h) &= 0 \text{ for } h < 0, \end{aligned} \quad (21)$$

where

$$f^*(t) = \begin{cases} A_1 \phi(t - \tau(t)) & t \in \mathcal{T} \\ 0 & \text{otherwise} \end{cases}.$$

A unique solution of non-homogeneous system (21) will take the form

$$x(t) = C(t, 0)x(0) + \int_0^t C(t, s) (Bf(s) + f^*(s)) ds, \quad (22)$$

The conditions on *delay-dependent positivity* [5]:

Lemma 3.2

The system (19) with $Bf(t) \geq 0$ for all $t \geq 0$, $x(0) \in \mathbb{R}_+^n$, with a Metzler matrix A_1 , $A_0 \geq 0$ and $0 \leq (A_0)_{i,i} \leq e(A_1)_{i,i} < (A_0)_{i,i} + \bar{\tau}^{-1}$ for all $i = 1, \dots, n$, has the corresponding solution $x(t) \geq 0$ for all $t \geq 0$ provided that

$$Bf(t) \geq -f^*(t) \quad \forall t \in [0, \bar{\tau}].$$

[5] Efimov et al. 2016.

Tools: *Another condition for positivity* [6]

Consider the second-order delay differential equation with an input signal:

$$\begin{aligned}\ddot{x}(t) &= -a(t)x(t-\tau) + b(t)x(t-\theta) + f(t) \quad t \in [0, +\infty), \\ x(h) &= \phi(h) \quad \text{for } h \leq 0, \quad \dot{x}(0) \in \mathbb{R}\end{aligned}\tag{23}$$

For a signal $q \in \mathcal{L}_\infty$ denote:

$$q_* = \operatorname{ess\,inf}_{t \geq 0} q(t), \quad q^* = \operatorname{ess\,sup}_{t \geq 0} q(t).$$

Theorem 3.1

Assume that $0 \leq \tau < \theta$ and there exists $\varepsilon > 0$ such that the inequalities

$$\varepsilon \leq \{a(t) - b(t)\} \leq \frac{1}{4} b_*^2 (\theta - \tau)^2, \quad \forall t \in [0, +\infty),\tag{24}$$

$$\frac{1}{\sqrt{a^*} \exp\left\{\frac{b_*(\theta-\tau)^2}{4}\right\}} \arctan \frac{b_*(\theta-\tau)}{2\sqrt{a^*} \exp\left\{\frac{b_*(\theta-\tau)^2}{4}\right\}} > \theta - \tau\tag{25}$$

are fulfilled. Then in a corresponding *homogeneous* equation for (23) the elements $C_{11}(t, s)$ and $C_{12}(t, s)$ of the Cauchy matrix $C(t, s) = \begin{bmatrix} C_{11}(t, s) & C_{12}(t, s) \\ C_{21}(t, s) & C_{22}(t, s) \end{bmatrix} \in \mathbb{R}^{2 \times 2}$ are nonnegative for $0 \leq s < t < +\infty$.

[6] Domoshnitsky, 2014.

Tools: *Extended condition for positivity*

Proposition 3.1

Let $0 < \tau < \theta$ in the system (23) and the conditions (24), (25) of the Theorem 3.1 hold. If $f(t) + f^*(t) \geq 0$ for all $t \geq 0$, $\dot{x}(0) \in \mathbb{R}_+$ and $\phi(h) \geq 0$ for all $h \in [-\theta, 0]$, where

$$f^*(t) = \begin{cases} -a(t)\phi(t-\tau) + b(t)\phi(t-\theta) & t \in [0, \tau] \\ b(t)\phi(t-\theta) & t \in (\tau, \theta], \\ 0 & t > \theta \end{cases}$$

then the corresponding solution satisfies $x(t) \geq 0$ for all $t \geq 0$.

Then the position solution for the system (23) will have all terms nonnegative:

$$x(t) = C_{11}(t, 0)x(0) + C_{12}(t, 0)\dot{x}(0) + \int_0^t C_{12}(t, s)(f(s) + f^*(s)) ds,$$

The first IO for the position $x(t)$

The first interval observer for the position:

$$\begin{aligned}\ddot{x}^-(t) &= -a(t)x^-(t-\tau) + b(t)x^-(t-\theta) \\ &\quad + \underline{f}(t) - b(t)y(t) - b(t)\nu_0 - \varrho(t), \quad (26) \\ \ddot{x}^+(t) &= -a(t)x^+(t-\tau) + b(t)x^+(t-\theta) \\ &\quad + \bar{f}(t) - b(t)y(t) + b(t)\nu_0 + \varrho(t),\end{aligned}$$

with initial conditions from Assumption 3.1:

$$\begin{aligned}x^-(h) &= \underline{\phi}(h), \quad x^+(h) = \bar{\phi}(h) \quad \forall h \in [-\theta, 0], \\ \dot{x}^-(0) &= \underline{\dot{x}}_0, \quad \dot{x}^+(0) = \bar{\dot{x}}_0,\end{aligned}$$

and a compensation term for $f^*(t)$

$$\varrho(t) = \begin{cases} a(t)[\bar{\phi}(t-\tau) - \underline{\phi}(t-\tau)] & t \leq \tau \\ 0 & t > \tau \end{cases}.$$

Proposition 3.2: Observer 1

Let the delays satisfy the relation $\theta \geq \tau > 0$, and assumptions 3.1, 3.2 be satisfied. For the system (17) with initial conditions (20) and the observer (26) select the observer gain $b(t) \geq 0$ to satisfy the conditions (24), (25) of Theorem 1 for all $t \geq 0$. Then its position satisfies the interval inclusion

$$x^-(t) \leq x(t) \leq x^+(t) \quad \forall t \in [0, +\infty) \quad (27)$$

and $x^+ - x, x - x^- \in \mathcal{L}_\infty$.

IO for the velocity $\dot{x}(t)$

Consider the negative delayed term $-a(t)x(t - \tau)$ as a disturbance and rewrite (19):

$$\dot{x}(t) = Ax(t) + \rho(t, x(t), x(t - \tau), f(t)), \quad (28)$$

where $x(t) = [x_1(t) \ x_2(t)]^T$ is the state vector, $x_1(t)$ is the estimated position by (27), $x_2(t)$ is a velocity of motion (17);

$$A = A_0 - KC = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix},$$

$$\rho(t, x(t), x(t - \tau), f(t)) = \begin{bmatrix} k_1 x_1(t) \\ k_2 x_1(t) + f(t) - a(t)x_1(t - \tau) \end{bmatrix},$$

$K = [k_1 \ k_2]^T$ is a *second observer gain*.

Assumption 3.3: Transformation

There are $K \in \mathbb{R}^2$ and a Metzler matrix $D \in \mathbb{R}^{2 \times 2}$ such that the matrices A and D have the same eigenvalues and the pairs (A, χ_1) and (D, χ_2) are observable for some $\chi_1 \in \mathbb{R}^{1 \times 2}$, $\chi_2 \in \mathbb{R}^{1 \times 2}$.

IO for the velocity $\dot{x}(t)$

New coordinates $z = Sx$ can be introduced transforming the system (28) to the form:

$$\dot{z}(t) = Dz(t) + S\rho(t, x(t), x(t - \tau), f(t)). \quad (29)$$

Lemma 3.3 (Efimov et al 2012)

For $A \in \mathbb{R}^{n \times m}$ and $x \in \mathbb{R}^n$ with $\underline{x} \leq x \leq \bar{x}$ for some $\underline{x}, \bar{x} \in \mathbb{R}^n$: $A^+ \underline{x} - A^- \bar{x} \leq Ax \leq A^+ \bar{x} - A^- \underline{x}$;

$$\underline{\rho}(t) \leq \rho(t, x(t), x(t - \tau), f(t)) \leq \bar{\rho}(t),$$

where the functions $\underline{\rho}(t)$ and $\bar{\rho}(t)$ depend only on available information (the variables $x_1^-(t)$, $x_1^+(t)$, $x_1^-(t - \tau)$ and $x_1^+(t - \tau)$ are given by the first observer (26) for all $t \in [0, +\infty)$):

$$\underline{\rho}(t) = \begin{bmatrix} k_1 x_1^-(t) \\ k_2 x_1^-(t) + \underline{f}(t) - a(t) x_1^+(t - \tau) \end{bmatrix},$$
$$\bar{\rho}(t) = \begin{bmatrix} k_1 x_1^+(t) \\ k_2 x_1^+(t) + \bar{f}(t) - a(t) x_1^-(t - \tau) \end{bmatrix}.$$

The second IO for the velocity $\dot{x}(t)$

The form of the interval observer for the velocity:

$$\begin{aligned}\dot{\underline{z}}(t) &= D\underline{z}(t) + S^+\underline{\rho}(t) - S^-\bar{\rho}(t), \\ \dot{\bar{z}}(t) &= D\bar{z}(t) + S^+\bar{\rho}(t) - S^-\underline{\rho}(t); \quad (30) \\ \underline{z}(h) &= S^+\underline{\Phi}(h) - S^-\bar{\Phi}(h), \\ \bar{z}(h) &= S^+\bar{\Phi}(h) - S^-\underline{\Phi}(h), \quad \forall h \in [-\tau, 0];\end{aligned}$$

The back transformation:

$$\begin{aligned}\underline{x}(t) &= (S^{-1})^+\underline{z}(t) - (S^{-1})^-\bar{z}(t), \\ \bar{x}(t) &= (S^{-1})^+\bar{z}(t) - (S^{-1})^-\underline{z}(t),\end{aligned}$$

where

$$\underline{\Phi}(h) = \begin{bmatrix} \underline{\phi}(h) \\ \begin{cases} 0 & h < 0 \\ \dot{\underline{x}}_0 & h = 0 \end{cases} \end{bmatrix}, \quad \bar{\Phi}(h) = \begin{bmatrix} \bar{\phi}(h) \\ \begin{cases} 0 & h < 0 \\ \bar{\dot{x}}_0 & h = 0 \end{cases} \end{bmatrix}$$

Proposition 3.3: Observer 2

Let assumptions 3.1, 3.2 and 3.3 be satisfied. Then for the second order delay equation (17), presented in form (19), with initial conditions (20), and with the interval observer (30) the relations for the velocity

$$\underline{x}_2(t) \leq x_2(t) \leq \bar{x}_2(t), \quad x(t) \in \mathbb{R}^2, \quad \forall t \in [0, +\infty) \quad (31)$$

are fulfilled provided that the conditions of Proposition 2 (*Observer 1*) are verified.

Simulation results

$$\ddot{x}(t) = -a(t)x(t - \tau) + f(t),$$

$$y(t) = x(t - \theta) + \nu(t).$$

Parameters for simulation:

- * $a = 2$, $\tau = 0.2$, $\theta = 0.5$,
- * $f(t) = 0.5(\cos(2t) + 0.3\cos(10t))$,
- * $\nu(t) = 0.07\sin(60t)$,
- * $x(h) = \phi(h) = 0.1\sin(\frac{1}{4\pi\theta}t)$, $\dot{x}(0) = 0$,

IO1 $b = 1.8$,

IO2 $K = [3 \ 1]$,
 $D = \text{diag}[-2.618; -0.382]$

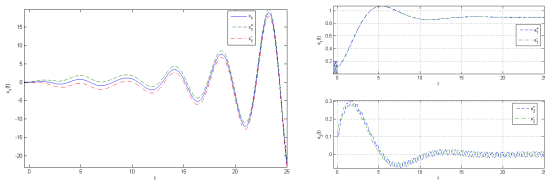


Figure 13: Simulation of the *first observer* (26) for the position $x(t)$ and the errors of estimation.

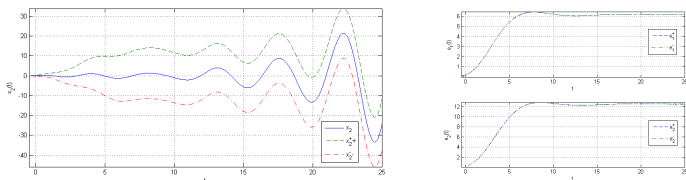


Figure 14: Simulation of the *second observer* (30) for the velocity $\dot{x}(t)$ and errors of estimation $e_1(t)$, $e_2(t)$.

Overall summary

- 1 An **interval observer** for an uncertain parabolic PDE with Dirichlet boundary conditions based on finite-element **Galerkin approximation**.
 - * The errors of discretization in the interval estimates.
 - * Stabilizing control of an uncertain PDE system based on interval observer.
 - * Experiments on Black-Scholes model.
- 2 A **PDE interval observer** for a non-homogeneous heat equation with Dirichlet boundary conditions.
 - * ISS stability of the observer depends on the sensors placing interval.
 - * Stabilization based on interval estimates.
 - * Comparison of the two approaches.
- 3 An **interval observer** approach for a second-order **delay differential equation** with delayed measurements and uncertainties.
 - * Extended conditions on positivity based on maximum principles.
 - * Two consecutively connected interval observers for the position and the velocity of the delayed motion model.

Journal Articles

★ **Automatica 93**, 2018

T. Kharkovskaia, D. Efimov, A. Polyakov, and J.-P. Richard. *Design of interval observers and controls for PDEs using finite-element approximations*

★ **Automatica 111**, 2020

T. Kharkovskaia, D. Efimov, E. Fridman, A. Polyakov, and J.-P. Richard. *Interval observer design and control of uncertain non-homogenous heat equations*

International Conferences

★ **NOLCOS16**, Monterey, California, USA

T. Kharkovskaia, D. Efimov, A. Polyakov, and J.-P. Richard. *Interval observers for PDEs: approximation approach*

★ **IFAC17**, Toulouse, France

T. Kharkovskaia, D. Efimov, E. Fridman, A. Polyakov, and J.-P. Richard. *On design of interval observers for parabolic PDEs*

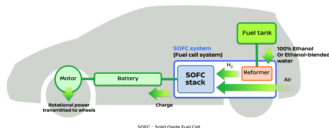
★ **CDC18**, Miami Beach, Florida, USA

T. Kharkovskaia, D. Efimov, E. Fridman, A. Polyakov, and J.-P. Richard. *Interval estimation for second-order delay differential equations with delayed measurements and uncertainties*

Directions of the future research

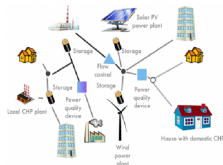
PDEs:

- Extend proposed interval estimation method for PDEs with Neumann, mixed boundary conditions;
- Study different types of PDEs: hyperbolic, nonlinear, etc;
- Implementation to a real world example (like a solid oxide fuel cell stack);
- Explore alternative ways for approximations of PDEs (Sturm-Liouville problem);



DDEs:

- Concerning delayed systems, extend the proposed algorithm to a more general LTV system;
- Chain of interval observer to cope with high value delays;
- Interval estimation for a plant with a network of interconnected distributed sensors;
- Interval observer for delayed PDEs, etc.



Thank You for your attention!

